

Krawtchouk polynomials, the Lie algebra \mathfrak{sl}_2 , and Leonard pairs

Kazumasa Nomura and Paul Terwilliger

January 10, 2012

Abstract

A Leonard pair is a pair of diagonalizable linear transformations of a finite-dimensional vector space, each of which acts in an irreducible tridiagonal fashion on an eigenbasis for the other one. In the present paper we give an elementary but comprehensive account of how the following are related: (i) Krawtchouk polynomials; (ii) finite-dimensional irreducible modules for the Lie algebra \mathfrak{sl}_2 ; (iii) a class of Leonard pairs said to have Krawtchouk type. Along the way we obtain elementary proofs of some well-known facts about Krawtchouk polynomials, such as the three-term recurrence, the orthogonality, the difference equation, and the generating function. The paper is a tutorial meant for a graduate student or a researcher unfamiliar with the above topics.

1 Introduction

This paper is about the relationship between the Krawtchouk polynomials, the Lie algebra \mathfrak{sl}_2 , and a class of Leonard pairs said to have Krawtchouk type. Before going into detail, we take a moment to establish some notation. Throughout the paper \mathbb{F} denotes a field. From now until the end of Section 4 we assume the characteristic $\text{Char}(\mathbb{F}) \neq 2$. Let N denote an integer. We now define what it means for N to be *feasible*. For the case $\text{Char}(\mathbb{F}) = 0$, N is feasible whenever $N \geq 0$. For the case $\text{Char}(\mathbb{F}) > 0$, N is feasible whenever $0 \leq N < \text{Char}(\mathbb{F})$. Let x denote an indeterminate and let $\mathbb{F}[x]$ denote the \mathbb{F} -algebra consisting of the polynomials in x that have all coefficients in \mathbb{F} . We now define some polynomials in $\mathbb{F}[x]$ called Krawtchouk polynomials [1, page 347], [15, Section 9.11]. Recall the shifted factorial

$$(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1), \quad n = 0, 1, 2, \dots$$

We interpret $(\alpha)_0 = 1$. By [1, Section 2.1] the ${}_2F_1$ hypergeometric series is

$${}_2F_1 \left(\begin{matrix} a & b \\ c \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

The Krawtchouk polynomials are defined using two parameters denoted N and p . The parameter N is a feasible integer and the parameter p is a scalar in \mathbb{F} such that $p \neq 0$ and

$p \neq 1$. For $i = 0, 1, \dots, N$ define a polynomial $K_i \in \mathbb{F}[x]$ by

$$K_i = K_i(x; p, N) = {}_2F_1 \left(\begin{matrix} -i & -x \\ -N \end{matrix} \middle| \frac{1}{p} \right). \quad (1)$$

We check that K_i is a well-defined polynomial in $\mathbb{F}[x]$. Observe that $(-i)_n$ vanishes for $n > i$, so in the hypergeometric series (1) the n -summand is zero for $n > i$. Also observe that $(-N)_n$ is nonzero for $n = 0, 1, \dots, i$. Therefore the n -summand in (1) has nonzero denominator for $n = 0, 1, \dots, i$. By these comments K_i is a well-defined polynomial in $\mathbb{F}[x]$. One checks that this polynomial has degree i , and the coefficient of x^i is $\frac{1}{(-N)_i p^i}$. The polynomial K_i is the i th Krawtchouk polynomial with parameters N and p . By the construction

$$K_i(j) = K_j(i), \quad i, j = 0, 1, \dots, N. \quad (2)$$

This is an example of a phenomenon known as self-duality [5] or more generally Askey-Wilson duality [3, Theorem 5.1], [27, Theorem 4.1].

We now recall the notion of a Leonard pair. A Leonard pair is a pair of diagonalizable linear transformations of a finite-dimensional vector space, each of which acts in an irreducible tridiagonal fashion on an eigenbasis for the other one [23, Definition 1.1]. For instance, for all feasible integers N the pair of matrices

$$S = \begin{pmatrix} 0 & 1 & & & \mathbf{0} \\ N & 0 & 2 & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & 2 & 0 & N \\ \mathbf{0} & & & & 1 & 0 \end{pmatrix}, \quad D = \text{diag}(N, N-2, \dots, -N)$$

acts on the vector space \mathbb{F}^{N+1} as a Leonard pair. To see this see [24, Section 1] or Lemma 4.8 below. This Leonard pair falls into a family said to have Krawtchouk type [27, Example 10.12]. See Definition 6.1 below for the definition of Krawtchouk type. See [13, Example 1.5], [28, Example 1.3] for more examples of Leonard pairs that have Krawtchouk type.

In the present paper we give an elementary but comprehensive account of how the following are related: (i) Krawtchouk polynomials; (ii) finite-dimensional irreducible \mathfrak{sl}_2 -modules; (iii) Leonard pairs of Krawtchouk type. The paper is a tutorial meant for a graduate student or a researcher unfamiliar with the above topics. In this regard the paper is similar to a paper of Junie Go [11] which provides an introduction to the subconstituent algebra [26] using the hypercube as a concrete example.

Before summarizing the present paper we briefly review the history concerning how the Krawtchouk polynomials are related to \mathfrak{sl}_2 . A relationship between the Krawtchouk polynomials and \mathfrak{sl}_2 was first given by Miller [19]; he observed that the difference equations for Krawtchouk polynomials come from the irreducible representations of \mathfrak{sl}_2 . Koornwinder [17, Section 2] observed that the matrix elements of a finite-dimensional irreducible representation of the group $SU(2)$ can be written in terms of Krawtchouk polynomials. This gives a connection between the Krawtchouk polynomials and \mathfrak{sl}_2 since the irreducible representations of $SU(2)$ and \mathfrak{sl}_2 are essentially the same. See [16, Section 2], [20, Sections 1,2], [29, Section 6.8.1] for more work on this topic. Later there appeared some articles

that gave a connection between Krawtchouk polynomials and \mathfrak{sl}_2 : [6], [7], [8, Section 4], [9, Chapter 5, IV]. In each of these articles, the above pair S, D acts as a bridge between \mathfrak{sl}_2 and Krawtchouk polynomials. On one hand, the matrix S (resp. D) represents the action of $e + f$ (resp. h) on the irreducible \mathfrak{sl}_2 -module with dimension $N + 1$. Here e, f, h denote the usual Chevalley basis for \mathfrak{sl}_2 . On the other hand S and D are related to the Krawtchouk polynomials $K_i(x; 1/2, N)$ in the following way. Sylvester [22] observed that the matrix S has eigenvalues $\{N - 2i\}_{i=0}^N$; this was recalled by Askey in [2, Section 1]. Since S has mutually distinct eigenvalues $\{N - 2i\}_{i=0}^N$, there exists an invertible matrix P such that $PSP^{-1} = D$. It turns out that, after a suitable normalization, the entries of P are

$$P_{ij} = \binom{N}{i} K_i(j; 1/2, N), \quad i, j = 0, 1, \dots, N.$$

As far as we know this fact was first observed by Kac [14, Section 4] in the context of probability theory. It later appeared in combinatorics, in the context of the Hamming association scheme [4, Theorem 4.2]; see also [21, Theorem 6] and [3, III.2].

We now summarize the contents of the present paper. We consider a type of element in \mathfrak{sl}_2 said to be normalized semisimple. Our main object of study is a pair a, a^* of normalized semisimple elements that generate \mathfrak{sl}_2 . We show that a, a^* satisfy a pair of relations

$$\begin{aligned} [a, [a, a^*]] &= 4(2p - 1)a + 4a^*, \\ [a^*, [a^*, a]] &= 4(2p - 1)a^* + 4a, \end{aligned}$$

where the scalar p depends on the \mathfrak{sl}_2 Killing form applied to a, a^* . The above equations are a special case of the Askey-Wilson relations [10, (3.2)], [28, Theorem 1.5]. We show that \mathfrak{sl}_2 has a presentation involving generators a, a^* subject to the above relations. We describe \mathfrak{sl}_2 from the point of view of this presentation. We show that \mathfrak{sl}_2 admits an antiautomorphism \dagger that fixes each of a, a^* . For all feasible integers N we consider an $(N + 1)$ -dimensional irreducible \mathfrak{sl}_2 -module V consisting of the homogeneous polynomials in two variables that have total degree N . We display a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on V such that $\langle \varphi.u, v \rangle = \langle u, \varphi^\dagger.v \rangle$ for all $\varphi \in \mathfrak{sl}_2$ and $u, v \in V$. We display two bases for V , denoted $\{v_i\}_{i=0}^N$ and $\{v_i^*\}_{i=0}^N$; the basis $\{v_i\}_{i=0}^N$ diagonalizes a and the basis $\{v_i^*\}_{i=0}^N$ diagonalizes a^* . We show that each of these bases is orthogonal with respect to $\langle \cdot, \cdot \rangle$. We show that

$$\langle v_i, v_j^* \rangle = K_i(j; p, N), \quad i, j = 0, 1, \dots, N.$$

Using these results we recover some well-known facts about Krawtchouk polynomials, such as the three-term recurrence, the orthogonality, the difference equation, and the generating function. We interpret these facts in terms of matrices. Finally we show that the pair a, a^* acts on the above \mathfrak{sl}_2 -module V as a Leonard pair of Krawtchouk type, and every Leonard pair of Krawtchouk type is obtained in this way.

The paper is organized as follows. In Section 2, after recalling some basic materials concerning \mathfrak{sl}_2 , we describe a pair of normalized semisimple elements that generate \mathfrak{sl}_2 . In Section 3 we describe how finite-dimensional irreducible \mathfrak{sl}_2 -modules look from the point of view of these elements. In this description we make heavy use of Krawtchouk polynomials. Along the way we recover some well-known facts about Krawtchouk polynomials. In Section 4 these facts are interpreted in terms of matrices. In Section 5 we bring in the notion of a Leonard pair. After obtaining some basic facts about general Leonard pairs, we focus on Leonard pairs of Krawtchouk type. In Section 6 we characterize Leonard pairs of Krawtchouk type as described in the last sentence of the previous paragraph.

2 The Lie algebra $\mathfrak{sl}_2(\mathbb{F})$

Throughout this section assume \mathbb{F} is algebraically closed. For all integers $n \geq 1$ let $\text{Mat}_n(\mathbb{F})$ denote the \mathbb{F} -algebra consisting of the $n \times n$ matrices that have all entries in \mathbb{F} .

The Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ consists of the matrices in $\text{Mat}_2(\mathbb{F})$ that have trace 0, together with the Lie bracket $[y, z] = yz - zy$. We abbreviate $L = \mathfrak{sl}_2(\mathbb{F})$. L has a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (3)$$

This basis satisfies

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \quad (4)$$

For $y \in L$ the *adjoint map* $\text{ady} : L \rightarrow L$ is the \mathbb{F} -linear transformation that sends $z \mapsto [y, z]$ for $z \in L$. The *Killing form* is the bilinear form $(\ , \) : L \times L \rightarrow \mathbb{F}$ such that $(y, z) = \text{tr}(\text{ady} \text{ ad}z)$ for $y, z \in L$, where tr means trace. For notational convenience define a bilinear form $\langle \ , \ \rangle : L \times L \rightarrow \mathbb{F}$ such that $\langle y, z \rangle = \frac{1}{8}(y, z)$ for $y, z \in L$. We abbreviate $\|y\|^2 = \langle y, y \rangle$. The values of $\langle \ , \ \rangle$ on the elements (3) are given as follows.

$\langle \ , \ \rangle$	e	h	f
e	0	0	$\frac{1}{2}$
h	0	1	0
f	$\frac{1}{2}$	0	0

(5)

Using (5) one finds $\langle y, z \rangle = \text{tr}(yz)/2$ for $y, z \in L$.

Pick $y \in L$ and write

$$y = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \beta e + \alpha h + \gamma f. \quad (6)$$

Then $\|y\|^2 = \alpha^2 + \beta\gamma = -\det(y)$. Let r, s denote the eigenvalues of y . Then $r + s = 0$ and $rs = -\|y\|^2$.

By an *automorphism* of L we mean an isomorphism of \mathbb{F} -vector spaces $\sigma : L \rightarrow L$ such that $[y, z]^\sigma = [y^\sigma, z^\sigma]$ for $y, z \in L$. Let σ denote an automorphism of L . Observe that $\text{ad}(y^\sigma) = \sigma(\text{ady})\sigma^{-1}$ for $y \in L$. Using this we find $\langle y, z \rangle = \langle y^\sigma, z^\sigma \rangle$ for $y, z \in L$. In particular $\|y\|^2 = \|y^\sigma\|^2$ for $y \in L$.

The following result is well-known [12, Section 2.3]; we give a short proof for the sake of completeness.

Lemma 2.1 [12, Section 2.3] *The following hold.*

- (i) *Let M denote an invertible matrix in $\text{Mat}_2(\mathbb{F})$. Then the map $L \rightarrow L$, $y \mapsto MyM^{-1}$ is an automorphism of L .*
- (ii) *Let σ denote an automorphism of L . Then there exists an invertible $M \in \text{Mat}_2(\mathbb{F})$ such that $y^\sigma = MyM^{-1}$ for $y \in L$.*

Proof. (i): Clear.

(ii): By (3) h has eigenvalues $1, -1$. Observe that $\|h^\sigma\|^2 = \|h\|^2$ so h^σ has eigenvalues $1, -1$. Therefore there exists an invertible $P \in \text{Mat}_2(\mathbb{F})$ such that $h^\sigma = PhP^{-1}$. Since the map $L \rightarrow L, y \mapsto PyP^{-1}$ is an automorphism of L , we may assume without loss of generality that $h^\sigma = h$. By (4) the element e is a basis for the eigenspace of $\text{ad}h$ associated with the eigenvalue 2. Applying σ and using $h^\sigma = h$ we see that e^σ is in this eigenspace. Therefore there exists $\beta \in \mathbb{F}$ such that $e^\sigma = \beta e$. Similarly there exists $\gamma \in \mathbb{F}$ such that $f^\sigma = \gamma f$. Using $[e^\sigma, f^\sigma] = h$ we find $\beta\gamma = 1$. Define the matrix $M = \text{diag}(\beta, 1)$. By the above comments $y^\sigma = MyM^{-1}$ for all $y \in \{e, h, f\}$. Therefore $y^\sigma = MyM^{-1}$ for all $y \in L$. \square

We recall a few definitions. Let V denote a nonzero finite-dimensional vector space over \mathbb{F} and let $A : V \rightarrow V$ denote an \mathbb{F} -linear transformation. We say that A is *diagonalizable* whenever V has a basis consisting of eigenvectors for A . Let $\{u_i\}_{i=1}^n$ denote a basis for V . For $B \in \text{Mat}_n(\mathbb{F})$ we say that B *represents A with respect to $\{u_i\}_{i=1}^n$* whenever $Au_j = \sum_{i=1}^n B_{ij}u_i$ for $j = 1, 2, \dots, n$.

Lemma 2.2 *Let $y \in L$ be as in (6). Then with respect to the basis e, h, f the matrix representing the \mathbb{F} -linear transformation $\text{ad}y : L \rightarrow L$ is*

$$\begin{pmatrix} 2\alpha & -2\beta & 0 \\ -\gamma & 0 & \beta \\ 0 & 2\gamma & -2\alpha \end{pmatrix}. \quad (7)$$

Proof. Use (4). \square

Corollary 2.3 *Let y denote an element of L and let $r, -r$ denote the eigenvalues of y . Then the eigenvalues of the \mathbb{F} -linear transformation $\text{ad}y : L \rightarrow L$ are $2r, 0, -2r$.*

Proof. Without loss we may assume that y is from (6), so that Lemma 2.2 applies. Compute the characteristic polynomial of (7) and simplify using $r^2 = \alpha^2 + \beta\gamma$. \square

An element $y \in L$ is said to be *semisimple* whenever the \mathbb{F} -linear transformation $\text{ad}y : L \rightarrow L$ is diagonalizable. Let σ denote an automorphism of L . Then y is semisimple if and only if y^σ is semisimple.

Let $0 \neq y \in L$ and let $r, -r$ denote the eigenvalues of y . We have two cases:

$$\text{Case } r = 0: \quad y^2 = 0, \quad \|y\|^2 = 0, \quad \det(y) = 0;$$

$$\text{Case } r \neq 0: \quad y \text{ is diagonalizable, } \|y\|^2 \neq 0, \quad \det(y) \neq 0.$$

The following result is well-known [12, Section 4.2]; we give a short proof for the sake of completeness.

Lemma 2.4 [12, Section 4.2] *For $y \in L$ the following are equivalent:*

- (i) y is semisimple.
- (ii) y is diagonalizable.

Proof. To avoid trivialities assume $y \neq 0$. Let $r, -r$ denote the eigenvalues of y . First assume that y is diagonalizable. We have $r \neq 0$ and $\text{Char}(\mathbb{F}) \neq 2$ so $2r, 0, -2r$ are mutually distinct. Now $\text{ad}y$ is diagonalizable so y is semisimple. Next assume that y is not diagonalizable. Then $r = 0$ so $\text{ad}y$ has all eigenvalues zero. The linear transformation $\text{ad}y$ is nonzero and nilpotent. Therefore $\text{ad}y$ is not diagonalizable so y is not semisimple. \square

For the following lemma the proof is routine and left to the reader.

Lemma 2.5 *For $y \in L$ the following are equivalent:*

- (i) $\|y\|^2 = 1$.
- (ii) $\det(y) = -1$.
- (iii) y is diagonalizable with eigenvalues $1, -1$.
- (iv) There exists an automorphism of L that sends $y \mapsto h$.

Given a semisimple $y \in L$, we say y is *normalized* whenever $\|y\|^2 = 1$.

Definition 2.6 For a pair a, a^* of normalized semisimple elements of L , we define $p \in \mathbb{F}$ such that $\langle a, a^* \rangle = 1 - 2p$. We call p the *corresponding parameter* for the pair a, a^* .

Example 2.7 Consider the pair of matrices

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}, \quad a^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (8)$$

where $\alpha^2 + \beta\gamma = 1$. Observe that each of a, a^* is a normalized semisimple element of L . For this pair the corresponding parameter p satisfies $p = (1 - \alpha)/2$ since $\langle a, a^* \rangle = \alpha$. Note that $\alpha = 1 - 2p$ and $\beta\gamma = 4p(1 - p)$.

Example 2.8 Consider the pair of matrices

$$a = \begin{pmatrix} 1 - 2p & 2(1 - p) \\ 2p & 2p - 1 \end{pmatrix}, \quad a^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (9)$$

where $p \in \mathbb{F}$. Then each of a, a^* is a normalized semisimple element of L , and p is the corresponding parameter. This is a special case of Example 2.7 with $\beta = 2(1 - p)$ and $\gamma = 2p$.

Lemma 2.9 *Let a, a^* denote a pair of normalized semisimple elements of L , and let p denote the corresponding parameter. Then the values of $\langle \cdot, \cdot \rangle$ on the elements $a, a^*, [a, a^*]$ are given as follows:*

$\langle \cdot, \cdot \rangle$	a	a^*	$[a, a^*]$
a	1	$1 - 2p$	0
a^*	$1 - 2p$	1	0
$[a, a^*]$	0	0	$-16p(1 - p)$

The above matrix has determinant $-64p^2(1 - p)^2$.

Proof. By Lemma 2.5(iv) we may assume without loss of generality that a, a^* are from Example 2.7. Observe that $[a, a^*] = -2\beta e + 2\gamma f$. Using this and (5) we routinely obtain the results. \square

Lemma 2.10 *Let a, a^* denote a pair of normalized semisimple elements of L , and let p denote the corresponding parameter. Then the following are equivalent:*

- (i) $p \neq 0$ and $p \neq 1$.
- (ii) The elements $a, a^*, [a, a^*]$ form a basis for L .
- (iii) The elements a, a^* generate L .

Proof. (i) \Rightarrow (ii): In Lemma 2.9 the matrix of inner products is nonsingular. Therefore the elements $a, a^*, [a, a^*]$ are linearly independent and hence form a basis for L .

(ii) \Rightarrow (i): The matrix (5) is nonsingular so the bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerate on L . Therefore the matrix in Lemma 2.9 has nonzero determinant.

(ii) \Rightarrow (iii): Clear.

(iii) \Rightarrow (ii): Observe $[a, a^*] \notin \text{Span}\{a, a^*\}$; otherwise $\text{Span}\{a, a^*\}$ is a Lie subalgebra of L which contradicts our assumption that a, a^* generate L . Note that a, a^* are linearly independent; otherwise $[a, a^*] = 0$. By these comments $a, a^*, [a, a^*]$ are linearly independent and hence form a basis for L . \square

Lemma 2.11 *Let a, a^* and b, b^* denote pairs of normalized semisimple elements of L , each of which generate L . Then the following are equivalent:*

- (i) $\langle a, a^* \rangle = \langle b, b^* \rangle$.
- (ii) There exists an automorphism of L that sends $a \mapsto b$ and $a^* \mapsto b^*$.

Proof. (i) \Rightarrow (ii): The pairs a, a^* and b, b^* have the same corresponding parameter, which we denote by p . Note that $p \neq 0, p \neq 1$ by Lemma 2.10. We first show that there exists an automorphism of L that sends a, a^* to the pair (9). By Lemma 2.5(iv) we may assume that a, a^* are as in (8). Note that $\gamma \neq 0$ since $\beta\gamma = 4p(1 - p)$ and $p \neq 0, p \neq 1$. Define $M = \text{diag}(1, 2p\gamma^{-1})$. Then M is invertible, and the automorphism $L \rightarrow L, y \mapsto MyM^{-1}$ sends a, a^* to the pair (9). Similarly there exists an automorphism of L that sends b, b^* to the pair (9). The result follows.

(ii) \Rightarrow (i): Clear. \square

Lemma 2.12 *Let a, a^* denote a pair of normalized semisimple elements of L , and let p denote the corresponding parameter. Then*

$$[a, [a, a^*]] = 4(2p - 1)a + 4a^*, \quad (10)$$

$$[a^*, [a^*, a]] = 4(2p - 1)a^* + 4a. \quad (11)$$

Proof. By Lemma 2.11 we may assume that a, a^* are as in (9). Using the matrices in (9) we routinely verify (10), (11). \square

Lemma 2.13 *Fix $p \in \mathbb{F}$ such that $p \neq 0$, $p \neq 1$. Let \mathcal{L} denote the Lie algebra over \mathbb{F} defined by generators u, v and relations*

$$[u, [u, v]] = 4(2p - 1)u + 4v, \quad (12)$$

$$[v, [v, u]] = 4(2p - 1)v + 4u. \quad (13)$$

Then \mathcal{L} is isomorphic to L . Moreover each of u, v is normalized semisimple, and p is the corresponding parameter.

Proof. Let the pair a, a^* be from Example 2.8. Note that a, a^* is a pair of normalized semisimple elements of L , and p is the corresponding parameter. We display a Lie algebra isomorphism $\mathcal{L} \rightarrow L$ that sends $u \mapsto a$ and $v \mapsto a^*$. By Lemma 2.12 the elements a, a^* satisfy (10), (11). Comparing these relations with (12), (13) we see that there exists a Lie algebra homomorphism $\varphi : \mathcal{L} \rightarrow L$ that sends $u \mapsto a$ and $v \mapsto a^*$. We show this homomorphism is bijective. By Lemma 2.10 the elements a, a^* generate L , so φ is surjective. Therefore $\dim \mathcal{L} \geq 3$. Using (12) and (13) we find \mathcal{L} is spanned by $u, v, [u, v]$. Therefore $\dim \mathcal{L} \leq 3$. By these comments $\dim \mathcal{L} = 3$ and φ is bijective. We have shown φ is an isomorphism of Lie algebras. The result follows. \square

Let a, a^* denote normalized semisimple elements that generate L , and let p denote the corresponding parameter. By Lemma 2.10 $p \neq 0$ and $p \neq 1$. By Lemma 2.11 there exists an automorphism of L that sends a, a^* to the pair (9). So without loss of generality we may assume a, a^* is the pair (9) with $p \neq 0$, $p \neq 1$. This assumption will be in effect until the end of Section 4. Thus

$$a = 2(1 - p)e + (1 - 2p)h + 2pf, \quad a^* = h. \quad (14)$$

Observe

$$[a, a^*] = 4(p - 1)e + 4pf. \quad (15)$$

By Lemma 2.10 the elements $a, a^*, [a, a^*]$ form a basis for L .

Lemma 2.14 *There exists a unique automorphism of L that sends $a \mapsto a^*$ and $a^* \mapsto a$. Denoting this automorphism by $*$ we have $(y^*)^* = y$ for $y \in L$.*

Proof. In Lemma 2.13 the relations are invariant under the map $u \mapsto v$, $v \mapsto u$. Therefore the automorphism exists. This automorphism is unique since a, a^* generate L . The last assertion is clear. \square

Let U, W denote the following matrices in $\text{Mat}_2(\mathbb{F})$:

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 1-p^{-1} \end{pmatrix}, \quad W = \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}. \quad (16)$$

One checks $WUWU = (1-p)I$. Define $R = WU$, so that $R^2 = (1-p)I$. We have

$$R = \begin{pmatrix} 1-p & 1-p \\ p & p-1 \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} 1 & 1 \\ \frac{p}{1-p} & -1 \end{pmatrix}. \quad (17)$$

Lemma 2.15 *For $y \in L$ we have $y^* = RyR^{-1}$.*

Proof. Observe that the map $L \rightarrow L$, $y \mapsto RyR^{-1}$ is an automorphism of L . Using (9) and (17) one checks $RaR^{-1} = a^*$ and $Ra^*R^{-1} = a$. The result follows in view of Lemma 2.14. \square

Recall that e, h, f is a basis for L . Applying the map $*$ to this basis we get another basis e^*, h^*, f^* for L . By (3), (4), (17) and Lemma 2.15,

$$e^* = (p-1)e + ph + \frac{p^2}{1-p}f, \quad (18)$$

$$h^* = 2(1-p)e + (1-2p)h + 2pf = a, \quad (19)$$

$$f^* = (1-p)e + (1-p)h + (p-1)f. \quad (20)$$

In summary we have the following three bases for L :

$$e, h, f; \quad a, a^*, [a, a^*]; \quad e^*, h^*, f^*. \quad (21)$$

We recall the notion of a transition matrix. Let V denote a nonzero finite-dimensional vector space over \mathbb{F} and let $\{u_i\}_{i=1}^n, \{v_i\}_{i=1}^n$ denote two bases for V . By the *transition matrix* from $\{u_i\}_{i=1}^n$ to $\{v_i\}_{i=1}^n$ we mean the matrix $T \in \text{Mat}_n(\mathbb{F})$ such that $v_j = \sum_{i=1}^n T_{ij}u_i$ for $j = 1, 2, \dots, n$. In this case T^{-1} exists, and equals the transition matrix from $\{v_i\}_{i=1}^n$ to $\{u_i\}_{i=1}^n$. Let $A : V \rightarrow V$ denote an \mathbb{F} -linear transformation and let $B \in \text{Mat}_n(\mathbb{F})$ denote the matrix that represents A with respect to $\{u_i\}_{i=1}^n$. Then the matrix $T^{-1}BT$ represents A with respect to $\{v_i\}_{i=1}^n$. Let $\{w_i\}_{i=1}^n$ denote a basis for V and let S denote the transition matrix from $\{v_i\}_{i=1}^n$ to $\{w_i\}_{i=1}^n$. Then TS is the transition matrix from $\{u_i\}_{i=1}^n$ to $\{w_i\}_{i=1}^n$.

Lemma 2.16 *For the bases (21) the transition matrices are given as follows:*

(i) *The transition matrix from the basis e, h, f to the basis $a, a^*, [a, a^*]$ is*

$$\begin{pmatrix} 2(1-p) & 0 & 4(p-1) \\ 1-2p & 1 & 0 \\ 2p & 0 & 4p \end{pmatrix},$$

and the transition matrix from the basis $a, a^, [a, a^*]$ to the basis e, h, f is*

$$\begin{pmatrix} \frac{1}{4(1-p)} & 0 & \frac{1}{4p} \\ \frac{2p-1}{4(1-p)} & 1 & \frac{2p-1}{4p} \\ \frac{1}{8(p-1)} & 0 & \frac{1}{8p} \end{pmatrix}.$$

(ii) The transition matrix from the basis e^*, h^*, f^* to the basis $a, a^*, [a, a^*]$ is

$$\begin{pmatrix} 0 & 2(1-p) & 4(1-p) \\ 1 & 1-2p & 0 \\ 0 & 2p & -4p \end{pmatrix},$$

and the transition matrix from the basis $a, a^*, [a, a^*]$ to the basis e^*, h^*, f^* is

$$\begin{pmatrix} \frac{2p-1}{4(1-p)} & 1 & \frac{2p-1}{4p} \\ \frac{1}{4(1-p)} & 0 & \frac{1}{4p} \\ \frac{1}{8(1-p)} & 0 & -\frac{1}{8p} \end{pmatrix}.$$

(iii) The transition matrix from the basis e, h, f to the basis e^*, h^*, f^* is

$$\begin{pmatrix} p-1 & 2(1-p) & 1-p \\ p & 1-2p & 1-p \\ \frac{p^2}{1-p} & 2p & p-1 \end{pmatrix},$$

and the transition matrix from the basis e^*, h^*, f^* to the basis e, h, f is

$$\begin{pmatrix} p-1 & 2(1-p) & 1-p \\ p & 1-2p & 1-p \\ \frac{p^2}{1-p} & 2p & p-1 \end{pmatrix}.$$

Proof. The first matrix of (i) follows from (14) and (15). To get the first matrix of (ii), apply the map $*$ to (14) and (15). The first matrix of (iii) follows from (18)–(20). Concerning the second matrix in (i)–(iii) just observe that it is the inverse of the first matrix. \square

Lemma 2.17 For each pair of bases among (21) the matrix representing $\langle \cdot, \cdot \rangle$ is given as follows:

$\langle \cdot, \cdot \rangle$	e	h	f
e	0	0	$\frac{1}{2}$
h	0	1	0
f	$\frac{1}{2}$	0	0

$\langle \cdot, \cdot \rangle$	a	a^*	$[a, a^*]$
a	1	$1-2p$	0
a^*	$1-2p$	1	0
$[a, a^*]$	0	0	$-16p(1-p)$

$\langle \cdot, \cdot \rangle$	e^*	h^*	f^*
e^*	0	0	$\frac{1}{2}$
h^*	0	1	0
f^*	$\frac{1}{2}$	0	0

$\langle \cdot, \cdot \rangle$	e^*	h^*	f^*
e	$\frac{p^2}{2(1-p)}$	p	$\frac{p-1}{2}$
h	p	$1-2p$	$1-p$
f	$\frac{p-1}{2}$	$1-p$	$\frac{1-p}{2}$

$\langle \cdot, \cdot \rangle$	a	a^*	$[a, a^*]$
e	p	0	$2p$
h	$1-2p$	1	0
f	$1-p$	0	$2(p-1)$

$\langle \cdot, \cdot \rangle$	a	a^*	$[a, a^*]$
e^*	0	p	$-2p$
h^*	1	$1-2p$	0
f^*	0	$1-p$	$2(1-p)$

Proof. The first table is from (5) and the second table is from Lemma 2.9. The remaining tables are obtained using (14), (15), (18)–(20). \square

Lemma 2.18 *Relative to each basis (21) the matrices representing ada , ada^* are given as follows:*

(i) *Relative to the basis $a, a^*, [a, a^*]$:*

$$\text{ada} : \begin{pmatrix} 0 & 0 & 4(2p-1) \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{ada}^* : \begin{pmatrix} 0 & 0 & -4 \\ 0 & 0 & 4(1-2p) \\ -1 & 0 & 0 \end{pmatrix}.$$

(ii) *Relative to the basis e, h, f :*

$$\text{ada} : \begin{pmatrix} 2(1-2p) & 4(p-1) & 0 \\ -2p & 0 & 2(1-p) \\ 0 & 4p & 2(2p-1) \end{pmatrix}, \quad \text{ada}^* : \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(iii) *Relative to the basis e^*, h^*, f^* :*

$$\text{ada} : \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \text{ada}^* : \begin{pmatrix} 2(1-2p) & 4(p-1) & 0 \\ -2p & 0 & 2(1-p) \\ 0 & 4p & 2(2p-1) \end{pmatrix}.$$

Proof. (i): The matrices are routinely obtained using (10) and (11).

(ii), (iii): Follows from (i) using Lemma 2.16 and the comments above Lemma 2.18.

\square

By an *antiautomorphism* of L we mean an isomorphism of \mathbb{F} -vector spaces $\sigma : L \rightarrow L$ such that $[y, z]^\sigma = [z^\sigma, y^\sigma]$ for $y, z \in L$.

Example 2.19 Each of the following maps is an antiautomorphism of L .

(i) The map $L \rightarrow L, y \mapsto -y$.

(ii) The map $L \rightarrow L, y \mapsto y^t$.

Consider two maps $\sigma : L \rightarrow L$ and $\tau : L \rightarrow L$, each of which is an automorphism or an antiautomorphism. If exactly one is an antiautomorphism, then the composition $\sigma\tau$ is an antiautomorphism. Otherwise $\sigma\tau$ is an automorphism.

Lemma 2.20 *The following hold.*

(i) *Let M denote an invertible matrix in $\text{Mat}_2(\mathbb{F})$. Then the map $L \rightarrow L, y \mapsto My^tM^{-1}$ is an antiautomorphism of L .*

(ii) *Let σ denote an antiautomorphism of L . Then there exists an invertible $M \in \text{Mat}_2(\mathbb{F})$ such that $y^\sigma = My^tM^{-1}$ for $y \in L$.*

Proof. (i): Follows from Lemma 2.1(i), Example 2.19(ii), and the comment below Example 2.19.

(ii): The map $L \rightarrow L$, $y \mapsto (y^t)^\sigma$ is an automorphism of L . So by Lemma 2.1(ii) there exists an invertible $M \in \text{Mat}_2(\mathbb{F})$ such that $(y^t)^\sigma = MyM^{-1}$ for $y \in L$. The result follows. \square

Lemma 2.21 *Let σ denote an antiautomorphism of L . Then $\langle y, z \rangle = \langle y^\sigma, z^\sigma \rangle$ for $y, z \in L$*

Proof. Define $\tau : L \rightarrow L$ such that $u^\tau = -u^\sigma$ for $u \in L$. Then τ is an automorphism of L . We have $\langle y, z \rangle = \langle y^\tau, z^\tau \rangle$, so $\langle y, z \rangle = \langle -y^\sigma, -z^\sigma \rangle = \langle y^\sigma, z^\sigma \rangle$. \square

Lemma 2.22 *There exists a unique antiautomorphism of L that fixes each of a, a^* . Denoting this antiautomorphism by \dagger we have $(y^\dagger)^\dagger = y$ for $y \in L$.*

Proof. Concerning existence, observe that the map $y \mapsto Wy^tW^{-1}$ is an antiautomorphism of L that fixes each of a, a^* , where W is from (16). We have shown \dagger exists. We now show that \dagger is unique. Let \dagger' denote an antiautomorphism of L that fixes each of a, a^* . We show that $\dagger' = \dagger$. The composition $\dagger\dagger'^{-1}$ is an automorphism of L that fixes each of a, a^* , so it must be the identity map since a, a^* generate L . So $\dagger = \dagger'$. Concerning the last assertion, observe the map $y \mapsto (y^\dagger)^\dagger$ is an automorphism of L that fixes each of a, a^* , and hence the identity map. \square

Lemma 2.23 *For $y \in L$ we have $y^\dagger = Wy^tW^{-1}$, where W is from (16).*

Proof. The map $y \mapsto Wy^tW^{-1}$ is an antiautomorphism that fixes each of a, a^* . By Lemma 2.22 such an antiautomorphism is unique. The result follows. \square

Lemma 2.24 *The maps $*$ and \dagger commute.*

Proof. For $y = a$ and $y = a^*$ we have $(y^*)^\dagger = (y^\dagger)^*$. \square

Lemma 2.25 *The antiautomorphism \dagger acts on e, h, f and e^*, h^*, f^* in the following way:*

$$\begin{array}{c|ccc|ccc} y & e & h & f & e^* & h^* & f^* \\ \hline y^\dagger & \frac{p}{1-p}f & h & \frac{1-p}{p}e & \frac{p}{1-p}f^* & h^* & \frac{1-p}{p}e^* \end{array}$$

Proof. $e^\dagger = We^tW^{-1} = \frac{p}{1-p}f$. The other cases are similar. \square

3 Krawtchouk polynomials and the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$

We continue to discuss the Lie algebra $L = \mathfrak{sl}_2(\mathbb{F})$. In this section we consider how L is related to Krawtchouk polynomials. We start by constructing a certain L -module. Let y, z denote commuting indeterminates. Let $\mathbb{F}[y, z]$ denote the \mathbb{F} -algebra consisting of the polynomials in y, z that have all coefficients in \mathbb{F} . We abbreviate $\mathcal{A} = \mathbb{F}[y, z]$. The \mathbb{F} -vector space \mathcal{A} has a basis

$$y^r z^s, \quad r, s = 0, 1, 2, \dots$$

For an integer $n \geq 0$ let $\text{Hom}_n(\mathcal{A})$ denote the n th homogeneous component of \mathcal{A} :

$$\text{Hom}_n(\mathcal{A}) = \text{Span} \{y^{n-i} z^i\}_{i=0}^n.$$

We abbreviate $H_n = \text{Hom}_n(\mathcal{A})$. Observe that $\dim H_n = n + 1$ and that $\mathcal{A} = \sum_{n=0}^{\infty} H_n$ (direct sum). Moreover $H_n H_m = H_{n+m}$ for $m, n \geq 0$. We have $H_0 = \mathbb{F}1$ and $H_1 = \mathbb{F}y + \mathbb{F}z$.

For a nonzero vector space V over \mathbb{F} , let $\text{End}(V)$ denote the \mathbb{F} -algebra consisting of all \mathbb{F} -linear transformations from V to V . Let $\mathfrak{gl}(V)$ denote the Lie algebra consisting of the \mathbb{F} -vector space $\text{End}(V)$ together with Lie bracket $[\varphi, \phi] = \varphi\phi - \phi\varphi$.

A *derivation* of \mathcal{A} is an element $\partial \in \mathfrak{gl}(\mathcal{A})$ such that $\partial(bc) = \partial(b)c + b\partial(c)$ for $b, c \in \mathcal{A}$. Let $\text{Der}(\mathcal{A})$ denote the set of all derivations of \mathcal{A} . One checks that $\text{Der}(\mathcal{A})$ is a Lie subalgebra of $\mathfrak{gl}(\mathcal{A})$. Observe that for $\partial \in \text{Der}(\mathcal{A})$,

$$\begin{aligned} \partial(1) &= 0, \\ \partial(b^n) &= nb^{n-1}\partial(b), & b \in \mathcal{A}, \quad n = 1, 2, \dots \\ \partial(y^r z^s) &= ry^{r-1}z^s\partial(y) + sy^r z^{s-1}\partial(z), & r, s = 0, 1, 2, \dots \end{aligned}$$

By these comments ∂ is determined by $\partial(y)$ and $\partial(z)$. Therefore ∂ is determined by its action on $\text{Hom}_1(\mathcal{A})$. We emphasize

$$\partial = 0 \text{ if and only if } \partial \text{ vanishes on } \text{Hom}_1(\mathcal{A}). \quad (22)$$

The following lemma asserts that any \mathbb{F} -linear transformation $\text{Hom}_1(\mathcal{A}) \rightarrow \mathcal{A}$ can be uniquely extended to $\text{Der}(\mathcal{A})$.

Lemma 3.1 *For an \mathbb{F} -linear transformation $\varphi : \text{Hom}_1(\mathcal{A}) \rightarrow \mathcal{A}$ there exists a unique $\partial = \partial_\varphi \in \text{Der}(\mathcal{A})$ such that the restriction of ∂ on $\text{Hom}_1(\mathcal{A})$ coincides with φ .*

Proof. There exists an element $\partial \in \mathfrak{gl}(\mathcal{A})$ such that

$$\partial(y^r z^s) = ry^{r-1}z^s\varphi(y) + sy^r z^{s-1}\varphi(z), \quad r, s = 0, 1, 2, \dots$$

One checks $\partial \in \text{Der}(\mathcal{A})$. By construction $\partial(y) = \varphi(y)$ and $\partial(z) = \varphi(z)$, so the restriction of ∂ on $\text{Hom}_1(\mathcal{A})$ coincides with φ . We have shown the existence of ∂ . The uniqueness follows from (22). \square

The Lie algebra L acts by left multiplication on the vector space \mathbb{F}^2 (column vectors). Recall that $\text{Hom}_1(\mathcal{A})$ has basis y, z . Consider the vector space isomorphism $\text{Hom}_1(\mathcal{A}) \rightarrow \mathbb{F}^2$

that sends $y \mapsto (1, 0)^t$ and $z \mapsto (0, 1)^t$. This isomorphism induces an L -module structure on $\text{Hom}_1(\mathcal{A})$ such that

$$\begin{aligned} e.y &= 0, & h.y &= y, & f.y &= z, \\ e.z &= y, & h.z &= -z, & f.z &= 0. \end{aligned} \quad (23)$$

Lemma 3.2 *The map $L \rightarrow \text{Der}(\mathcal{A})$, $\varphi \mapsto \partial_\varphi$ is an injective homomorphism of Lie algebras.*

Proof. We first show that the map is a homomorphism of Lie algebras. It suffices to show

$$\partial_{[\varphi, \phi]} = [\partial_\varphi, \partial_\phi] \quad (\varphi, \phi \in L). \quad (24)$$

In the equation (24) both sides are contained in $\text{Der}(\mathcal{A})$ and they agree on $\text{Hom}_1(\mathcal{A})$. So this equation holds in view of (22). Therefore the map is a homomorphism of Lie algebras. The injectivity is clear by construction. \square

We have proven the following theorem.

Theorem 3.3 *The algebra \mathcal{A} has an L -module structure such that each element of L acts on \mathcal{A} as a derivation and (23) holds.*

For the rest of this section we fix a feasible integer N . We consider the subspace $\text{Hom}_N(\mathcal{A})$ of \mathcal{A} . This subspace has a basis $\{y^{N-i}z^i\}_{i=0}^N$.

Lemma 3.4 *The elements e, h, f act on the basis $\{y^{N-i}z^i\}_{i=0}^N$ as follows:*

$$\begin{aligned} e.(y^{N-i}z^i) &= iy^{N-i+1}z^{i-1} & (1 \leq i \leq N), & & e.y^N &= 0, \\ h.(y^{N-i}z^i) &= (N-2i)y^{N-i}z^i & (0 \leq i \leq N), & & & \\ f.(y^{N-i}z^i) &= (N-i)y^{N-i-1}z^{i+1} & (0 \leq i \leq N-1), & & f.z^N &= 0. \end{aligned}$$

Proof. The element e acts on \mathcal{A} as a derivation, so for $i = 0, 1, \dots, N$,

$$e.(y^{N-i}z^i) = (N-i)y^{N-i-1}z^i(e.y) + iy^{N-i}z^{i-1}(e.z).$$

In this equation the right-hand side is equal to $iy^{N-i+1}z^{i-1}$ in view of (23). The other cases are similar. \square

The following lemma is a reformulation of Lemma 3.4.

Lemma 3.5 *With respect to the basis $\{y^{N-i}z^i\}_{i=0}^N$ the matrices representing e, h, f are*

$$\begin{aligned} e : & \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 0 \\ 0 & & & & N \\ & & & & 0 \end{pmatrix}, & f : & \begin{pmatrix} 0 & & & & 0 \\ N & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 2 & 0 \\ 0 & & & 1 & 0 \end{pmatrix}, \\ & & h : & \text{diag}(N, N-2, \dots, -N). \end{aligned}$$

Lemmas 3.4 or 3.5 shows that $\text{Hom}_N(\mathcal{A})$ is an L -submodule of \mathcal{A} . One checks that this L -module is irreducible. If $\text{Char}(\mathbb{F}) = 0$ then up to isomorphism $\text{Hom}_N(\mathcal{A})$ is the unique

irreducible L -module of dimension $N + 1$ [12, Theorem 7.2]. More generally we have the following.

Lemma 3.6 *Let V denote an irreducible L -module with dimension $N + 1$. Then the following are equivalent.*

- (i) *The L -module V is isomorphic to $\text{Hom}_N(\mathcal{A})$.*
- (ii) *V has a basis $\{v_i\}_{i=0}^N$ such that $h.v_i = (N - 2i)v_i$ for $i = 0, 1, \dots, N$ and both*

$$e.v_0 = 0, \quad f.v_N = 0.$$

Proof. (i) \Rightarrow (ii): Immediate from Lemma 3.4.

(ii) \Rightarrow (i): For $i = 0, 1, \dots, N$ the vector v_i is an eigenvector for h with eigenvalue $N - 2i$. Note that $\{N - 2i\}_{i=0}^N$ are mutually distinct. Pick any integer i ($1 \leq i \leq N$). Using $[h, e] = 2e$ we find $e.v_i \in \mathbb{F}v_{i-1}$, and using $[h, f] = -2f$ we find $f.v_{i-1} \in \mathbb{F}v_i$. Define $\alpha_i, \beta_i \in \mathbb{F}$ such that $e.v_i = \alpha_i v_{i-1}$ and $f.v_{i-1} = \beta_i v_i$. Define $\gamma_i = \alpha_i \beta_i$. For $i = 0, 1, \dots, N$ apply each side of $[e, f] = h$ to v_i and find $\gamma_{i+1} - \gamma_i = N - 2i$, where $\gamma_0 = 0$ and $\gamma_{N+1} = 0$. Solving this recursion we obtain $\gamma_i = i(N - i + 1)$ for $i = 0, 1, \dots, N$. Renormalizing the basis $\{v_i\}_{i=0}^N$ we may assume $\alpha_i = i$ and $\beta_i = N - i + 1$ for $1 \leq i \leq N$. Now with respect to $\{v_i\}_{i=0}^N$ the matrices representing e, f, h match those from Lemma 3.5. Therefore there exists an isomorphism of L -modules $V \rightarrow \text{Hom}_N(\mathcal{A})$ that sends $v_i \mapsto y^{N-i}z^i$ for $i = 0, 1, \dots, N$. \square

For the rest of this section we abbreviate $V = \text{Hom}_N(\mathcal{A})$. For $i = 0, 1, \dots, N$ define $V_i = \mathbb{F}y^{N-i}z^i$. We have $\dim V_i = 1$ and

$$V = \sum_{i=0}^N V_i \quad (\text{direct sum}). \quad (25)$$

For $i = 0, 1, \dots, N$ the space V_i is the eigenspace of h associated with the eigenvalue $N - 2i$. We call V_i the h -weight space for the eigenvalue $N - 2i$. We call (25) the h -weight space decomposition of V .

Recall the basis e^*, h^*, f^* for L from (18)–(20). We now describe the action of e^*, h^*, f^* on the L -module V . We will use the matrix R from (17). Recall that y, z form a basis for $\text{Hom}_1(\mathcal{A})$. Define

$$y^* = (1 - p)y + pz, \quad z^* = (1 - p)y + (p - 1)z. \quad (26)$$

Then y^*, z^* form a basis for $\text{Hom}_1(\mathcal{A})$, and R is the transition matrix from y, z to y^*, z^* . We have

$$y = y^* + \frac{p}{1 - p}z^*, \quad z = y^* - z^*. \quad (27)$$

Lemma 3.7 *The elements e^*, h^*, f^* act on $\text{Hom}_1(\mathcal{A})$ as follows:*

$$\begin{aligned} e^*.y^* &= 0, & h^*.y^* &= y^*, & f^*.y^* &= z^*, \\ e^*.z^* &= y^*, & h^*.z^* &= -z^*, & f^*.z^* &= 0. \end{aligned} \quad (28)$$

Proof. Use (18)–(20), (23), and (26). \square

By construction $\{y^{*N-i}z^{*i}\}_{i=0}^N$ form a basis for V .

Lemma 3.8 *The elements e^*, h^*, f^* act on the basis $\{y^{*N-i}z^{*i}\}_{i=0}^N$ as follows:*

$$\begin{aligned} e^*. (y^{*N-i}z^{*i}) &= iy^{*N-i+1}z^{*i-1} & (1 \leq i \leq N), & & e^*.y^{*N} &= 0, \\ h^*. (y^{*N-i}z^{*i}) &= (N-2i)y^{*N-i}z^{*i} & (0 \leq i \leq N), & & & \\ f^*. (y^{*N-i}z^{*i}) &= (N-i)y^{*N-i-1}z^{*i+1} & (0 \leq i \leq N-1), & & f^*.z^{*N} &= 0. \end{aligned}$$

Proof. Similar to the proof of Lemma 3.4 using (28). \square

The following lemma is a reformulation of Lemma 3.8.

Lemma 3.9 *With respect to the basis $\{y^{*N-i}z^{*i}\}_{i=0}^N$ the matrices representing e^*, h^*, f^* are*

$$\begin{aligned} e^* : \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & N \\ 0 & & & & 0 \end{pmatrix}, & \quad f^* : \begin{pmatrix} 0 & & & & 0 \\ N & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 2 & 0 \\ 0 & & & & 1 & 0 \end{pmatrix}, \\ h^* : \text{diag}(N, N-2, \dots, -N). \end{aligned}$$

For $i = 0, 1, \dots, N$ define $V_i^* = \mathbb{F}y^{*N-i}z^{*i}$. We have $\dim V_i^* = 1$ and

$$V = \sum_{i=0}^N V_i^* \quad (\text{direct sum}). \quad (29)$$

For $i = 0, 1, \dots, N$ the space V_i^* is the eigenspace of h^* associated with the eigenvalue $N - 2i$. We call V_i^* the h^* -weight space for the eigenvalue $N - 2i$. We call (29) the h^* -weight space decomposition of V .

Definition 3.10 For notational convenience define

$$k_i = \binom{N}{i} \left(\frac{p}{1-p} \right)^i \quad i = 0, 1, \dots, N. \quad (30)$$

Note that $k_0 = 1$.

We now define a bilinear form $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$. As we will see, both

$$\langle V_i, V_j \rangle = 0 \quad \text{if } i \neq j, \quad i, j = 0, 1, \dots, N, \quad (31)$$

$$\langle V_i^*, V_j^* \rangle = 0 \quad \text{if } i \neq j, \quad i, j = 0, 1, \dots, N. \quad (32)$$

Definition 3.11 Define a bilinear form $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$ by

$$\langle y^{N-i}z^i, y^{N-j}z^j \rangle = \delta_{i,j} \frac{1}{k_i(1-p)^N}, \quad i, j = 0, 1, \dots, N \quad (33)$$

where $\{k_i\}_{i=0}^N$ are from Definition 3.10. Observe that $\langle \cdot, \cdot \rangle$ is symmetric, nondegenerate, and satisfies (31).

Lemma 3.12 For $\varphi \in L$ and $u, v \in V$ we have

$$\langle \varphi.u, v \rangle = \langle u, \varphi^\dagger.v \rangle,$$

where \dagger is the antiautomorphism of L from Lemma 2.22.

Proof. Without loss of generality, we assume φ is in the basis e, h, f and u, v are in the basis $\{y^{N-i}z^i\}_{i=0}^N$. Write $u = y^{N-i}z^i$ and $v = y^{N-j}z^j$. First assume that $\varphi = e$. Using Lemma 3.4 and (33),

$$\langle e.u, v \rangle = \langle e.(y^{N-i}z^i), y^{N-j}z^j \rangle = \langle iy^{N-i+1}z^{i-1}, y^{N-j}z^j \rangle = i\delta_{i-1,j} \frac{1}{k_j(1-p)^N}.$$

By Lemma 2.25 $e^\dagger = \frac{p}{1-p}f$. Now using Lemma 3.4 and (33),

$$\langle u, e^\dagger.v \rangle = \left\langle y^{N-i}z^i, \frac{p(N-j)}{1-p}y^{N-j-1}z^{j+1} \right\rangle = \frac{p(N-j)}{1-p} \delta_{i-1,j} \frac{1}{k_j(1-p)^N}.$$

By (30) we have $(1-p)ik_i = p(N-j)k_j$ provided $i-1 = j$. By these comments $\langle e.u, v \rangle = \langle u, e^\dagger.v \rangle$. The proof is similar for the case $\varphi = h$ or $\varphi = f$. \square

Lemma 3.13 The bilinear form $\langle \cdot, \cdot \rangle$ satisfies (32).

Proof. Let i, j be given with $i \neq j$. Pick $u \in V_i^*$ and $v \in V_j^*$, so that $h^*.u = (N-2i)u$ and $h^*.v = (N-2j)v$. Observe

$$(N-2i)\langle u, v \rangle = \langle h^*.u, v \rangle = \langle u, (h^*)^\dagger.v \rangle = \langle u, h^*.v \rangle = (N-2j)\langle u, v \rangle.$$

By assumption $\text{Char}(\mathbb{F}) \neq 2$. Also since N is feasible, $\text{Char}(\mathbb{F})$ is 0 or greater than N . Therefore $2i \neq 2j$. By these comments $\langle u, v \rangle = 0$. \square

Given a basis $\{u_i\}_{i=0}^N$ for V , there exists a unique basis $\{v_i\}_{i=0}^N$ for V such that $\langle u_i, v_i \rangle = \delta_{i,j}$ for $i, j = 0, 1, \dots, N$. The bases $\{u_i\}_{i=0}^N$ and $\{v_i\}_{i=0}^N$ are said to be *dual* with respect to $\langle \cdot, \cdot \rangle$.

Lemma 3.14 With respect to $\langle \cdot, \cdot \rangle$ the basis for V dual to $\{y^{N-i}z^i\}_{i=0}^N$ is $\{k_i(1-p)^N y^{N-i}z^i\}_{i=0}^N$.

Proof. Immediate from (33). \square

Lemma 3.15 For the dual basis in Lemma 3.14 the sum of the basis vectors is y^{*N} .

Proof. Using (26) and (30),

$$\sum_{i=0}^N k_i (1-p)^N y^{N-i} z^i = \sum_{i=0}^N \binom{N}{i} (1-p)^{N-i} p^i y^{N-i} z^i = ((1-p)y + pz)^N = y^{*N}.$$

□

Lemma 3.16 For $i, j = 0, 1, \dots, N$,

$$\langle y^{*N-i} z^{*i}, y^{*N-j} z^{*j} \rangle = \delta_{i,j} k_i^{-1}. \quad (34)$$

Proof. We assume $i = j$; otherwise (34) holds by Lemma 3.13. We proceed using induction on i . First assume that $i = 0$. Observe

$$\begin{aligned} \|y^{*N}\|^2 &= \left\| \sum_{\ell=0}^N k_\ell (1-p)^N y^{N-\ell} z^\ell \right\|^2 && \text{(by Lemma 3.15)} \\ &= \sum_{\ell=0}^N \left\| k_\ell (1-p)^N y^{N-\ell} z^\ell \right\|^2 && \text{(by (31))} \\ &= \sum_{\ell=0}^N k_\ell (1-p)^N && \text{(by (33))} \\ &= \sum_{\ell=0}^N \binom{N}{\ell} (1-p)^{N-\ell} p^\ell && \text{(by (30))} \\ &= (1-p+p)^N \\ &= 1. \end{aligned}$$

Therefore (34) holds for $i = 0$. Next assume that $i \geq 1$. By Lemma 3.12,

$$\langle e^* \cdot (y^{*N-i} z^{*i}), y^{*N-i+1} z^{*i-1} \rangle = \langle y^{*N-i} z^{*i}, (e^*)^\dagger \cdot (y^{*N-i+1} z^{*i-1}) \rangle. \quad (35)$$

By Lemma 3.8 the left-hand side of (35) is equal to $\|y^{*N-i+1} z^{*i-1}\|^2 i$ and this is equal to ik_{i-1}^{-1} by induction. By Lemmas 2.25 and 3.8, the right-hand side of (35) is equal to $\|y^{*N-i} z^{*i}\|^2 (N-i+1)p(1-p)^{-1}$. By these comments $ik_{i-1}^{-1} = \|y^{*N-i} z^{*i}\|^2 (N-i+1)p(1-p)^{-1}$. Evaluating this using (30) we obtain $\|y^{*N-i} z^{*i}\|^2 = k_i^{-1}$. Therefore (34) holds at i and the result follows. □

Lemma 3.17 With respect to $\langle \cdot, \cdot \rangle$ the basis for V dual to $\{y^{*N-i} z^{*i}\}_{i=0}^N$ is $\{k_i y^{*N-i} z^{*i}\}_{i=0}^N$.

Proof. Immediate from Lemma 3.16. □

Lemma 3.18 *For the dual basis in Lemma 3.17 the sum of the basis vectors is y^N .*

Proof. Using (27) and (30),

$$\sum_{i=0}^N k_i y^{*N-i} z^{*i} = \sum_{i=0}^N \binom{N}{i} \left(\frac{p}{1-p} \right)^i y^{*N-i} z^{*i} = \left(y^* + \frac{p}{1-p} z^* \right)^N = y^N.$$

□

We have been discussing the bases $\{y^{N-i} z^i\}_{i=0}^N$ and $\{y^{*N-i} z^{*i}\}_{i=0}^N$ for V . We now find the transition matrices between these bases. We use Krawtchouk polynomials $\{K_i(x)\}_{i=0}^N$ from (1).

Lemma 3.19 *For $j = 0, 1, \dots, N$ both*

$$y^{*N-j} z^{*j} = \sum_{i=0}^N \binom{N}{i} (1-p)^{N-i} p^i K_i(j) y^{N-i} z^i, \quad (36)$$

$$y^{N-j} z^j = \sum_{i=0}^N \binom{N}{i} \left(\frac{p}{1-p} \right)^i K_i(j) y^{*N-i} z^{*i}. \quad (37)$$

Proof. We first show (36). By (27) we have $z^* = y^* - z$, so the left-hand side of (36) is

$$y^{*N-j} (y^* - z)^j = y^{*N-j} \sum_{\ell=0}^j \binom{j}{\ell} (-1)^\ell y^{*j-\ell} z^\ell = \sum_{\ell=0}^N \frac{(-j)_\ell}{\ell!} y^{*N-\ell} z^\ell.$$

By (1) the right-hand side of (36) is

$$\sum_{\ell=0}^N \frac{(-j)_\ell}{(-N)_\ell \ell! p^\ell} \sum_{i=0}^N \binom{N}{i} (1-p)^{N-i} p^i (-i)_\ell y^{N-i} z^i.$$

So it suffices to show that for $\ell = 0, 1, \dots, N$,

$$y^{*N-\ell} z^\ell = \frac{1}{(-N)_\ell p^\ell} \sum_{i=0}^N \binom{N}{i} (1-p)^{N-i} p^i (-i)_\ell y^{N-i} z^i. \quad (38)$$

In the right-hand side of (38) the i th term vanishes for $i < \ell$. So changing the variable

$r = i - \ell$, the right-hand side of (38) becomes

$$\begin{aligned}
& \frac{1}{(-N)_\ell p^\ell} \sum_{r=0}^{N-\ell} \binom{N}{r+\ell} (1-p)^{N-r-\ell} p^{r+\ell} (-r-\ell)_\ell y^{N-r-\ell} z^{r+\ell} \\
&= \frac{(-1)^\ell (N-\ell)!}{N!} \sum_{r=0}^{N-\ell} \frac{N!}{(r+\ell)!(N-r-\ell)!} (1-p)^{N-r-\ell} p^r \frac{(-1)^\ell (r+\ell)!}{r!} y^{N-r-\ell} z^{r+\ell} \\
&= z^\ell \sum_{r=0}^{N-\ell} \frac{(N-\ell)!}{r!(N-\ell-r)!} (1-p)^{N-\ell-r} p^r y^{N-\ell-r} z^r \\
&= z^\ell \sum_{r=0}^{N-\ell} \binom{N-\ell}{r} ((1-p)y)^{N-\ell-r} (pz)^r \\
&= z^\ell ((1-p)y + pz)^{N-\ell} \\
&= z^\ell y^{*N-\ell} \quad (\text{by (26)}).
\end{aligned}$$

Thus (38) holds. We have shown (36). The proof of (37) is similar. \square

We now find the inner products between the bases $\{y^{N-i} z^i\}_{i=0}^N$ and $\{y^{*N-i} z^{*i}\}_{i=0}^N$.

Lemma 3.20 For $i, j = 0, 1, \dots, N$,

$$\langle y^{N-i} z^i, y^{*N-j} z^{*j} \rangle = K_i(j).$$

Proof. Use (30), (33), and (36). \square

Define \mathbb{F} -linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ by

$$A = \frac{NI - a}{2}, \quad A^* = \frac{NI - a^*}{2}, \quad (39)$$

where a, a^* are from (14). Note that on V ,

$$a = NI - 2A, \quad a^* = NI - 2A^*. \quad (40)$$

Theorem 3.21 For $j = 0, 1, \dots, N$ both

$$K_j(A) y^N = y^{N-j} z^j, \quad (41)$$

$$K_j(A^*) y^{*N} = y^{*N-j} z^{*j}. \quad (42)$$

Proof. We first show (41). By (19) we have $a = h^*$. By this and Lemma 3.8, for $i = 0, 1, \dots, N$ the vector $y^{*N-i} z^{*i}$ is an eigenvector for a with eigenvalue $N - 2i$. Therefore $y^{*N-i} z^{*i}$ is an eigenvector for A with eigenvalue i . Now using Lemma 3.18 along with (30), (37) we obtain

$$K_j(A) y^N = K_j(A) \sum_{i=0}^N y^{*N-i} z^{*i} k_i = \sum_{i=0}^N y^{*N-i} z^{*i} k_i K_j(i) = y^{N-j} z^j.$$

We have shown (41). The proof of (42) is similar. \square

For the rest of this section, we use our results so far to easily recover some well-known properties of Krawtchouk polynomials.

Theorem 3.22 [15, Section 9.11] *Krawtchouk polynomials satisfy the following orthogonality relations:*

(i) For $i, j = 0, 1, \dots, N$,

$$\sum_{n=0}^N K_n(i) K_n(j) \binom{N}{n} p^n (1-p)^{N-n} = \delta_{i,j} \binom{N}{i}^{-1} \left(\frac{1-p}{p} \right)^i. \quad (43)$$

(ii) For $n, m = 0, 1, \dots, N$,

$$\sum_{i=0}^N K_n(i) K_m(i) \binom{N}{i} p^i (1-p)^{N-i} = \delta_{n,m} \binom{N}{n}^{-1} \left(\frac{1-p}{p} \right)^n. \quad (44)$$

Proof. (i): We compute $\langle y^{*N-i} z^{*i}, y^{*N-j} z^{*j} \rangle$ in two ways. On one hand, by Lemma 3.16 and (30) we find that $\langle y^{*N-i} z^{*i}, y^{*N-j} z^{*j} \rangle$ is equal to the right-hand side of (43). On the other hand, by (36),

$$y^{*N-i} z^{*i} = \sum_{n=0}^N \binom{N}{n} (1-p)^{N-n} p^n K_n(i) y^{N-n} z^n, \quad (45)$$

$$y^{*N-j} z^{*j} = \sum_{m=0}^N \binom{N}{m} (1-p)^{N-m} p^m K_m(j) y^{N-m} z^m. \quad (46)$$

Computing $\langle y^{*N-i} z^{*i}, y^{*N-j} z^{*j} \rangle$ using (45), (46) and Definition 3.11 we find it equals the left-hand side of (43). Therefore (43) holds.

(ii): Follows from (i) using (2). \square

Krawtchouk polynomials satisfy the following three-term recurrence.

Theorem 3.23 [15, Section 9.11] *For $i, x = 0, 1, \dots, N$,*

$$x K_i(x) = i(p-1) K_{i-1}(x) - (i(p-1) + (i-N)p) K_i(x) + (i-N)p K_{i+1}(x). \quad (47)$$

Proof. By Lemma 3.12 and $h^\dagger = h$,

$$\langle h.(y^{N-x} z^x), y^{*N-i} z^{*i} \rangle = \langle y^{N-x} z^x, h.(y^{*N-i} z^{*i}) \rangle. \quad (48)$$

We first evaluate the left-hand side of (48). To do this use Lemmas 3.4 and 3.20. We now evaluate the right-hand side of (48). By Lemma 2.16(iii),

$$h = 2(1-p)e^* + (1-2p)h^* + 2pf^*. \quad (49)$$

Evaluate the right-hand side of (48) using (49), and simplify the result using Lemmas 3.8, 3.20. By these comments (48) reduces to

$$(N - 2x)K_i(x) = 2(1 - p)iK_{i-1}(x) + (1 - 2p)(N - 2i)K_i(x) + 2p(N - i)K_{i+1}(x).$$

In this equation we rearrange terms to get (47). \square

Krawtchouk polynomials satisfy the following difference equation.

Theorem 3.24 [15, Section 9.11] *For $i, x = 0, 1, \dots, N$,*

$$iK_i(x) = x(p - 1)K_i(x - 1) - (x(p - 1) + (x - N)p)K_i(x) + (x - N)pK_i(x + 1). \quad (50)$$

Proof. In (47) exchange i and x , and use (2). \square

Krawtchouk polynomials have the following generating function.

Theorem 3.25 [15, Section 9.11] *Let t denote an indeterminate. Then for $x = 0, 1, \dots, N$,*

$$\left(1 - \frac{1 - p}{p}t\right)^x (1 + t)^{N - x} = \sum_{i=0}^N \binom{N}{i} K_i(x) t^i. \quad (51)$$

Proof. We apply (36) with $y = \frac{1}{1 - p}$ and $z = \frac{t}{p}$. Using (26) we find $y^* = 1 + t$ and $z^* = 1 - \frac{1 - p}{p}t$. The result follows. \square

4 Description by matrices

In Section 3 we used a certain L -module V to describe Krawtchouk polynomials $\{K_i(x)\}_{i=0}^N$. In this section we summarize our results in matrix form.

We comment on the notation. Recall that $\text{Mat}_{N+1}(\mathbb{F})$ denotes the \mathbb{F} -algebra consisting of all $(N + 1) \times (N + 1)$ matrices with entries in \mathbb{F} . From now on, we adopt the convention that for each matrix in this algebra the rows and columns are indexed by $0, 1, \dots, N$.

Definition 4.1 Define matrices U, B, D, K in $\text{Mat}_{N+1}(\mathbb{F})$ as follows. For $i, j = 0, 1, \dots, N$ the (i, j) -entry of U is $K_i(j)$. The matrix B is tridiagonal:

$$B = \begin{pmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & c_2 & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ \mathbf{0} & & & \cdot & \cdot & b_{N-1} \\ & & & c_N & a_N \end{pmatrix},$$

where

$$c_i = i(p - 1), \quad b_i = (i - N)p, \quad a_i = -b_i - c_i. \quad (52)$$

The matrix D is diagonal with (i, i) -entry i for $i = 0, 1, \dots, N$. The matrix K is diagonal with (i, i) -entry k_i for $i = 0, 1, \dots, N$, where $\{k_i\}_{i=0}^N$ are from (30).

Note 4.2 Referring to (30) and (52),

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}, \quad i = 0, 1, \dots, N. \quad (53)$$

Theorem 4.3 *With reference to Definition 4.1 the following hold.*

- (i) $U^t = U$.
- (ii) $B^t = KBK^{-1}$.
- (iii) $UD = BU$.
- (iv) $DU = UB^t$.
- (v) $(1 - p)^N UKUK = I$.

Proof. (i): By (2).

(ii): One routinely checks $KB = B^t K$ by matrix multiplication, using the tridiagonal shape of B and (53).

(iii): This is the three-term recurrence (47) in matrix form.

(iv): This is the difference equation (50) in matrix form.

(v): This is the orthogonality relation (44) in matrix form. \square

Sometimes it is convenient to work with the following matrix.

Definition 4.4 Define $P = UK$ where the matrices U, K are from Definition 4.1.

Theorem 4.3 looks as follows in terms of P .

Theorem 4.5 *With reference to Definitions 4.1 and 4.4 the following hold:*

- (i) $P^t = KPK^{-1}$.
- (ii) $B^t = KBK^{-1}$.
- (iii) $PD = BP$.
- (iv) $PB = DP$.
- (v) $P^2 = (1 - p)^{-N} I$.

Proof. In Theorem 4.3 eliminate U using $U = PK^{-1}$. \square

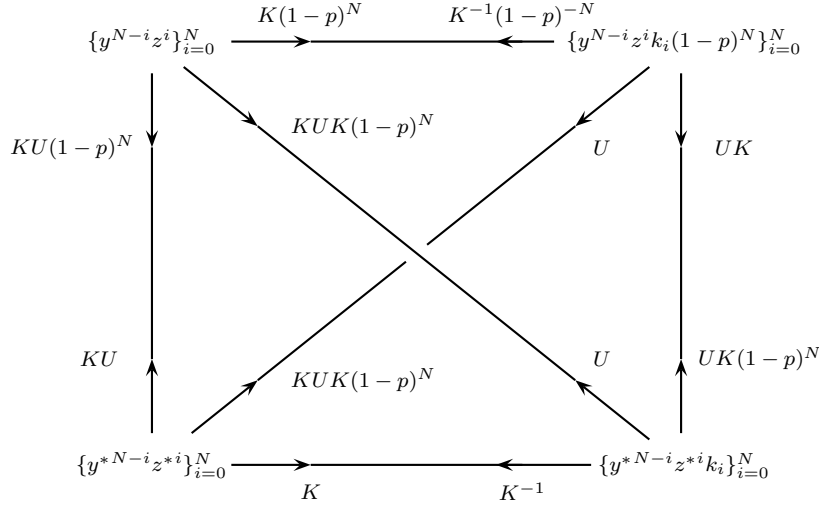
In Section 3 we encountered the following bases for V :

$$\{y^{N-i} z^i\}_{i=0}^N \quad \{y^{N-i} z^i k_i (1 - p)^N\}_{i=0}^N \quad (54)$$

$$\{y^{*N-i} z^{*i}\}_{i=0}^N \quad \{y^{*N-i} z^{*i} k_i\}_{i=0}^N \quad (55)$$

On each line (54), (55) the two bases on that line are dual with respect to \langle , \rangle . We now give the transition matrices between the four bases in (54), (55).

Lemma 4.6 *In the diagram below we display the transition matrices between the four bases in (54), (55):*



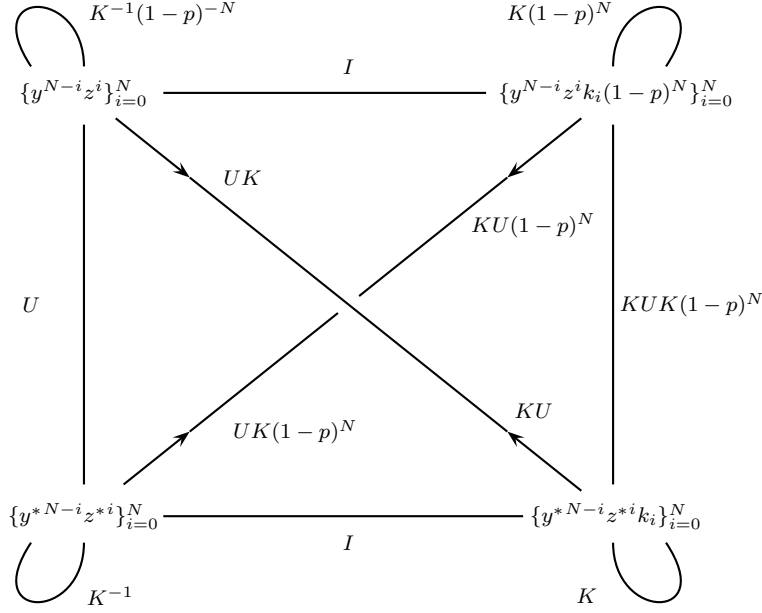
Transition matrices

$$\{u_i\}_{i=0}^N \xrightarrow{M} \{v_i\}_{i=0}^N \text{ means } v_j = \sum_{i=0}^N M_{ij} u_i \quad (0 \leq j \leq N)$$

Proof. To get the transition matrices along the left vertical line, reformulate (36) and (37) using (30) and Definition 4.1. The transition matrices along the two horizontal lines are immediate from Definition 4.1. The remaining matrices are obtained using the comments above Lemma 2.16. \square

We now give the inner products between the four bases in (54), (55).

Lemma 4.7 *In the diagram below we display the inner products between the four bases in (54), (55):*



Inner products

$$\{u_i\}_{i=0}^N \xrightarrow{M} \{v_i\}_{i=0}^N \text{ means } M_{ij} = \langle u_i, v_j \rangle \quad (0 \leq i, j \leq N)$$

The direction arrow is left off if M is symmetric

Proof. Follows from Definition 3.11 and Lemmas 3.16, 3.20 using Definition 4.1. \square

Recall the linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ from (39).

Lemma 4.8 *With respect to our four bases in (54), (55) the matrices representing A and A^* are given in the table below:*

basis	$\{y^{N-i}z^i\}_{i=0}^N$	$\{y^{N-i}z^i k_i (1-p)^N\}_{i=0}^N$	$\{y^{*N-i}z^{*i} k_i\}_{i=0}^N$	$\{y^{*N-i}z^{*i}\}_{i=0}^N$
A	B^t	B	D	D
A^*	D	D	B	B^t

Proof. Use (14), (39) and Lemmas 3.5, 3.9 along with $B^t = KBK^{-1}$. \square

We now summarize the essential relationship between A and A^* . We will use the following notion. A tridiagonal matrix is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. For example the tridiagonal matrix B from Definition 4.1 is irreducible. Now consider the bases $\{y^{N-i}z^i\}_{i=0}^N$ and $\{y^{*N-i}z^{*i}\}_{i=0}^N$ for V . With respect to these bases the matrices representing A and A^* take the following form:

basis	matrix representing A	matrix representing A^*
$\{y^{N-i}z^i\}_{i=0}^N$	irreducible tridiagonal	diagonal
$\{y^{*N-i}z^{*i}\}_{i=0}^N$	diagonal	irreducible tridiagonal

In Sections 5 and 6 we investigate this relationship in a more abstract setting, using the notion of a Leonard pair.

5 Leonard pairs

In [18] Doug Leonard characterized a family of orthogonal polynomials consisting of the q -Racah polynomials and their relatives. This family is sometimes called the terminating branch of the Askey scheme [15], [27, Section 24]. In [23] the second author introduced the notion of a Leonard pair in order to clarify and simplify Leonard's characterization. We now define a Leonard pair.

Throughout this section $\text{Char}(\mathbb{F})$ will be arbitrary. Let V denote a vector space over \mathbb{F} with finite positive dimension.

Definition 5.1 [23, Definition 1.1] By a *Leonard pair* on V we mean an ordered pair of \mathbb{F} -linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ that satisfy (i) and (ii) below.

- (i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
- (ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.

By the *diameter* of the above Leonard pair we mean the dimension of V minus one.

Note 5.2 Let A, A^* denote a Leonard pair on V . Then A^*, A is a Leonard pair on V . Also for $\alpha, \alpha^*, \beta, \beta^* \in \mathbb{F}$ with $\alpha\alpha^* \neq 0$, the pair $\alpha A + \beta I, \alpha^* A^* + \beta^* I$ is a Leonard pair on V .

The Leonard pairs are classified up to isomorphism [23, 27]. By that classification there is a natural correspondence between the Leonard pairs and the orthogonal polynomials that make up the terminating branch of the Askey scheme. Krawtchouk polynomials are members of the terminating branch of the Askey scheme. Our next general goal is to characterize the Leonard pairs that correspond to Krawtchouk polynomials.

An element $A \in \text{End}(V)$ is said to be *multiplicity-free* whenever A is diagonalizable and each eigenspace of A has dimension one.

Lemma 5.3 *Let A, A^* denote a Leonard pair on V . Then each of A, A^* is multiplicity-free.*

Proof. Concerning A , by Definition 5.1(ii) there exists a basis for V consisting of eigenvectors for A . Therefore A is diagonalizable. We now show that each eigenspace of A has dimension one. To this end, we show that the number of the eigenspaces of A is equal to the dimension of V . Note that the number of eigenspaces of A is equal to the degree of the minimal polynomial of A . We now find this degree. By Definition 5.1(i) there exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal; denote this matrix by B . By construction A, B have the same minimal polynomial. By the irreducible tridiagonal shape of B we find that I, B, B^2, \dots, B^N are linearly independent, where $N = \dim V - 1$. Therefore the minimal polynomial of B has degree $N + 1 = \dim V$.

By these comments the degree of the minimal polynomial of A is equal to the dimension of V . Consequently each eigenspace of A has dimension one, so A is multiplicity-free. The case of A^* is similar. \square

When working with a Leonard pair, it is often convenient to consider a closely related object called a Leonard system. In order to define this we first recall some concepts from linear algebra. For the rest of this section set $N = \dim V - 1$. Let A denote a multiplicity-free element of $\text{End}(V)$, and let $\{\theta_i\}_{i=0}^N$ denote an ordering of the eigenvalues of A . For $i = 0, 1, \dots, N$ let V_i denote the eigenspace of A associated with θ_i . So

$$V = \sum_{i=0}^N V_i \quad (\text{direct sum}).$$

For $i = 0, 1, \dots, N$ define $E_i \in \text{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ if $j \neq i$ ($j = 0, 1, \dots, N$). We call E_i the *primitive idempotent* of A associated with θ_i . Observe (i) $E_i E_j = \delta_{i,j} E_i$ ($i, j = 0, 1, \dots, N$); (ii) $I = \sum_{i=0}^N E_i$; (iii) $A = \sum_{i=0}^N \theta_i E_i$; (iv) $V_i = E_i V$ ($i = 0, 1, \dots, N$). Moreover

$$E_i = \prod_{\substack{0 \leq j \leq N \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j}, \quad i = 0, 1, \dots, N. \quad (56)$$

Let \mathcal{D} denote the \mathbb{F} -subalgebra of $\text{End}(V)$ generated by A . Observe that each of $\{A^i\}_{i=0}^N$ and $\{E_i\}_{i=0}^N$ is a basis for \mathcal{D} . Moreover $\prod_{i=0}^N (A - \theta_i I) = 0$. Note that $\text{tr}(E_i) = 1$ and $\text{rank}(E_i) = 1$ for $i = 0, 1, \dots, N$.

Definition 5.4 [23, Definition 1.4] By a *Leonard system* on V we mean a sequence

$$\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$$

such that

- (i) Each of A, A^* is a multiplicity-free element of $\text{End}(V)$.
- (ii) $\{E_i\}_{i=0}^N$ is an ordering of the primitive idempotents of A .
- (iii) $\{E_i^*\}_{i=0}^N$ is an ordering of the primitive idempotents of A^* .
- (iv) $E_i A^* E_j = \begin{cases} 0 & \text{if } |i - j| > 1, \\ \neq 0 & \text{if } |i - j| = 1, \end{cases} \quad i, j = 0, 1, \dots, N.$
- (v) $E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i - j| > 1, \\ \neq 0 & \text{if } |i - j| = 1, \end{cases} \quad i, j = 0, 1, \dots, N.$

Leonard pairs and Leonard systems are related as follows. Let A, A^* denote a Leonard pair on V . Let $\{w_i\}_{i=0}^N$ denote a basis for V from Definition 5.1(ii), and let $\{w_i^*\}_{i=0}^N$ denote a basis for V from Definition 5.1(i). Each w_i is an eigenvector for A ; let E_i denote the corresponding primitive idempotent of A . Each w_i^* is an eigenvector for A^* ; let E_i^* denote the corresponding primitive idempotent of A^* . Then $(A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ is a Leonard

system on V . Conversely, let $\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ denote a Leonard system on V . For $i = 0, 1, \dots, N$ pick nonzero vectors $w_i \in E_i V$ and $w_i^* \in E_i^* V$. Then $\{w_i\}_{i=0}^N$ is a basis for V that satisfies Definition 5.1(ii), and $\{w_i^*\}_{i=0}^N$ is a basis for V that satisfies Definition 5.1(i). Therefore A, A^* is a Leonard pair on V . We say the Leonard pair A, A^* and the Leonard system Φ are *associated*.

Let $\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ denote a Leonard system on V . Using Φ there are several ways to get another Leonard system on V . For instance, let $\alpha, \alpha^*, \beta, \beta^*$ denote scalars in \mathbb{F} with $\alpha\alpha^* \neq 0$. Then

$$(\alpha A + \beta I, \{E_i\}_{i=0}^N, \alpha^* A^* + \beta^* I, \{E_i^*\}_{i=0}^N)$$

is a Leonard system on V . Also each of the following is a Leonard system on V :

$$\begin{aligned}\Phi^* &= (A^*, \{E_i^*\}_{i=0}^N, A, \{E_i\}_{i=0}^N), \\ \Phi^\downarrow &= (A, \{E_{N-i}\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N), \\ \Phi^\downarrow^* &= (A, \{E_i\}_{i=0}^N, A^*, \{E_{N-i}^*\}_{i=0}^N).\end{aligned}$$

Let A, A^* denote a Leonard pair and let Φ denote an associated Leonard system. Then A, A^* is associated with $\Phi, \Phi^\downarrow, \Phi^\downarrow^*, \Phi^*$, and no other Leonard system.

Definition 5.5 [23, Definition 1.8] Let $\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ denote a Leonard system on V . For $i = 0, 1, \dots, N$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*). We call $\{\theta_i\}_{i=0}^N$ (resp. $\{\theta_i^*\}_{i=0}^N$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ .

Definition 5.6 [25, Definition 7.2] Let A, A^* denote a Leonard pair. By an *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of A, A^* we mean the eigenvalue sequence (resp. dual eigenvalue sequence) of an associated Leonard system.

Note 5.7 Let A, A^* denote a Leonard pair, with eigenvalue sequence $\{\theta_i\}_{i=0}^N$. Then $\{\theta_{N-i}\}_{i=0}^N$ is an eigenvalue sequence of A, A^* and A, A^* has no further eigenvalue sequence. A similar comment applies to dual eigenvalue sequences.

Definition 5.8 Let $\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ denote a Leonard system on V . For $i = 0, 1, \dots, N$ pick a nonzero $v_i \in E_i^* V$, and note that $\{v_i\}_{i=0}^N$ is a basis for V . We call such a basis a Φ -*basis* for V .

Proposition 5.9 Let $(A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ denote a Leonard system on V , and let \mathcal{D} denote the \mathbb{F} -subalgebra of $\text{End}(V)$ generated by A . Then the \mathbb{F} -linear transformation $\mathcal{D} \otimes \mathcal{D} \rightarrow \text{End}(V)$, $x \otimes y \mapsto xE_0^*y$ is an isomorphism of \mathbb{F} -vector spaces.

Proof. Recall that $\{A^i\}_{i=0}^N$ form a basis for \mathcal{D} , so it suffices to show that the elements

$$A^r E_0^* A^s, \quad r, s = 0, 1, \dots, N \quad (57)$$

form a basis for $\text{End}(V)$. Let $\{v_i\}_{i=0}^N$ denote a Φ -basis for V . Identify each element of $\text{End}(V)$ with the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents it with respect to $\{v_i\}_{i=0}^N$. From

this point of view A is an irreducible tridiagonal matrix and $E_0^* = \text{diag}(1, 0, \dots, 0)$. For $r, s = 0, 1, \dots, N$ we show that the entries of $A^r E_0^* A^s$ satisfy

$$(A^r E_0^* A^s)_{ij} = \begin{cases} 0 & \text{if } i > r \text{ or } j > s, \\ \neq 0 & \text{if } i = r \text{ and } j = s, \end{cases} \quad i, j = 0, 1, \dots, N. \quad (58)$$

By matrix multiplication,

$$(A^r E_0^* A^s)_{ij} = (A^r)_{i0} (A^s)_{0j}, \quad i, j = 0, 1, \dots, N. \quad (59)$$

Using the irreducible tridiagonal shape of A we find that for $i = 0, 1, \dots, N$ the entry $(A^r)_{i0}$ is zero if $i > r$, and nonzero if $i = r$. Similarly for $j = 0, 1, \dots, N$ the entry $(A^s)_{0j}$ is zero if $j > s$, and nonzero if $j = s$. Combining these facts with (59) we routinely obtain (58). Therefore the elements (57) are linearly independent. The number of elements in (57) is $(N+1)^2$ and this is equal to the dimension of $\text{End}(V)$. By these comments the elements (57) form a basis for $\text{End}(V)$. The result follows. \square

Corollary 5.10 *Let $(A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ denote a Leonard system on V . Then the elements*

$$E_i E_0^* E_j, \quad i, j = 0, 1, \dots, N \quad (60)$$

form a basis for $\text{End}(V)$.

Proof. By Proposition 5.9 and since $\{E_i\}_{i=0}^N$ form a basis for \mathcal{D} . \square

Lemma 5.11 *Let $(A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ denote a Leonard system on V . Then the following hold for $r = 0, 1, \dots, N$.*

$$(i) \quad E_i^* A^r E_j^* = \begin{cases} 0 & \text{if } r < |i - j|, \\ \neq 0 & \text{if } r = |i - j|, \end{cases} \quad i, j = 0, 1, \dots, N.$$

$$(ii) \quad E_i A^{*r} E_j = \begin{cases} 0 & \text{if } r < |i - j|, \\ \neq 0 & \text{if } r = |i - j|, \end{cases} \quad i, j = 0, 1, \dots, N.$$

Proof. (i): Fix a Φ -basis $\{v_i\}_{i=0}^N$ for V . Identify each element of $\text{End}(V)$ with the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents it with respect to $\{v_i\}_{i=0}^N$. Using this point of view the result is routinely obtained.

(ii): Similar to the proof of (i). \square

For the rest of this section fix a Leonard system $\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ on V , with eigenvalue sequence $\{\theta_i\}_{i=0}^N$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^N$. Define

$$a_i = \text{tr}(A E_i^*), \quad a_i^* = \text{tr}(A^* E_i), \quad i = 0, 1, \dots, N. \quad (61)$$

Lemma 5.12 *The following hold for $i = 0, 1, \dots, N$.*

- (i) $E_i^* A E_i^* = a_i E_i^*$.
- (ii) $E_i A^* E_i = a_i^* E_i$.

Proof. (i): Fix a Φ -basis $\{v_i\}_{i=0}^N$ for V . Identify each element of $\text{End}(V)$ with the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents it with respect to $\{v_i\}_{i=0}^N$. Using this point of view the result is routinely obtained.

(ii): Similar to the proof of (i). \square

Lemma 5.13 *Assume $N \geq 1$. Then the following hold.*

$$A^* A E_0^* = \theta_1^* A E_0^* + a_0(\theta_0^* - \theta_1^*) E_0^*, \quad (62)$$

$$A^* A E_N^* = \theta_{N-1}^* A E_N^* + a_N(\theta_N^* - \theta_{N-1}^*) E_N^*, \quad (63)$$

$$E_0 A^* A = \theta_1 E_0 A^* + a_0^*(\theta_0 - \theta_1) E_0, \quad (64)$$

$$E_N A^* A = \theta_{N-1} E_N A^* + a_N^*(\theta_N - \theta_{N-1}) E_N. \quad (65)$$

Proof. We first show (62). Using $I = \sum_{i=0}^N E_i^*$ and Definition 5.4(v) we find $A E_0^* = I A E_0^* = E_0^* A E_0^* + E_1^* A E_0^*$. By this and Lemma 5.12(i),

$$A E_0^* = a_0 E_0^* + E_1^* A E_0^*. \quad (66)$$

In equation (66), multiply each side on the left by $A^* - \theta_1^* I$, and simplify the result using $A^* E_0^* = \theta_0^* E_0^*$ and $A^* E_1^* = \theta_1^* E_1^*$. This yields (62). Applying (62) to Φ^\downarrow we get (63). The proofs of (64) and (65) are similar. \square

Lemma 5.14 *Assume $N \geq 1$. Then the following hold.*

- (i) $E_0 A^* A E_0^* = ((a_0 - \theta_0)(\theta_0^* - \theta_1^*) + \theta_0 \theta_0^*) E_0 E_0^*$.
- (ii) $E_0 A^* A^2 E_0^* = (\theta_0 + \theta_1) E_0 A^* A E_0^* - \theta_0 \theta_1 \theta_0^* E_0 E_0^*$.

Proof. (i): In equation (62), multiply each side on the left by E_0 and use $E_0 A = \theta_0 E_0$ to obtain the result.

(ii): In equation (64), multiply each side on the right by $A - \theta_0 I$. Simplify the result using $E_0 A = \theta_0 E_0$ to find

$$E_0 A^* A^2 = (\theta_0 + \theta_1) E_0 A^* A - \theta_0 \theta_1 E_0 A^*. \quad (67)$$

In equation (67), multiply each side on the right by E_0^* , and simplify the result using $A^* E_0^* = \theta_0^* E_0^*$. The result follows. \square

Lemma 5.15 *Assume $N \geq 1$. Then the following hold.*

- (i) $E_N^* A A^* E_N = ((a_N^* - \theta_N^*)(\theta_N - \theta_{N-1}) + \theta_N \theta_N^*) E_N^* E_N$.
- (ii) $E_N^* A^2 A^* E_N = (\theta_{N-1} + \theta_N) E_N^* A A^* E_N - \theta_{N-1} \theta_N \theta_N^* E_N^* E_N$.

Proof. (i): Apply Lemma 5.14(i) to $\Phi^{*\downarrow\downarrow}$.

(ii): Similar to the proof of Lemma 5.14(ii). \square

Lemma 5.16 *Assume $N \geq 1$. Then*

$$a_0(\theta_0^* - \theta_1^*) + a_N^*(\theta_{N-1} - \theta_N) = \theta_0^*\theta_{N-1} - \theta_1^*\theta_N. \quad (68)$$

Proof. Let α denote the left-hand side of (68) minus the right-hand side of (68). We show $\alpha = 0$. Consider the expression which is E_N times (62) minus (65) times E_0^* . Simplifying this expression using $E_N A = \theta_N E_N$ and $A^* E_0^* = \theta_0^* E_0^*$ we get $\alpha E_N E_0^* = 0$. Note that $E_N E_0^* \neq 0$ by Corollary 5.10 so $\alpha = 0$. The result follows. \square

6 Leonard pairs of Krawtchouk type

Our discussion of $\mathfrak{sl}_2(\mathbb{F})$ in Sections 2–4 was under the assumption that \mathbb{F} is algebraically closed with $\text{Char}(\mathbb{F}) \neq 2$. Once again we make this assumption. For the rest of the paper fix a feasible integer N .

Definition 6.1 Let A, A^* denote a Leonard pair of diameter N . This Leonard pair is said to have *Krawtchouk type* whenever $\{N - 2i\}_{i=0}^N$ is both an eigenvalue sequence and dual eigenvalue sequence of A, A^* .

Lemma 6.2 *Let V denote a vector space over \mathbb{F} with dimension $N+1$ and let A, A^* denote a Leonard pair on V . Let $\{\theta_i\}_{i=0}^N$ (resp. $\{\theta_i^*\}_{i=0}^N$) denote an eigenvalue sequence (resp. dual eigenvalue sequence) of A, A^* . Then the following are equivalent:*

- (i) *Each of $\{\theta_i\}_{i=0}^N$ and $\{\theta_i^*\}_{i=0}^N$ is an arithmetic progression.*
- (ii) *There exist scalars $\alpha, \alpha^*, \beta, \beta^*$ in \mathbb{F} with $\alpha\alpha^* \neq 0$ such that the Leonard pair $\alpha A + \beta I, \alpha^* A^* + \beta^* I$ has Krawtchouk type.*

Proof. Routine. \square

In the following two theorems we characterize the Leonard pairs of Krawtchouk type using $L = \mathfrak{sl}_2(\mathbb{F})$.

Theorem 6.3 *Consider the L -module $V = \text{Hom}_N(A)$ from Section 3. Let a, a^* denote normalized semisimple elements that generate L . Then a, a^* act on V as a Leonard pair of Krawtchouk type.*

Proof. By the comment above (14) we may assume that the basis e, h, f for L is related to a, a^* according to (14). Consider the elements $A, A^* \in \text{End}(V)$ from (39). Recall the basis $\{y^{N-i}z^i\}_{i=0}^N$ for V from above Lemma 3.4 and the basis $\{y^{*N-i}z^{*i}\}_{i=0}^N$ for V from above Lemma 3.8. By Lemma 4.8 the action of A, A^* on these bases is described as follows. With respect to the basis $\{y^{N-i}z^i\}_{i=0}^N$ the matrix representing A is irreducible tridiagonal and the matrix representing A^* is $\text{diag}(0, 1, \dots, N)$. With respect to the basis $\{y^{*N-i}z^{*i}\}_{i=0}^N$ the matrix representing A is $\text{diag}(0, 1, \dots, N)$ and the matrix representing A^* is irreducible tridiagonal. Now by (40), with respect to the basis $\{y^{N-i}z^i\}_{i=0}^N$ the matrix representing a is irreducible tridiagonal and the matrix representing a^* is $\text{diag}(N, N-2, \dots, -N)$. Moreover, with respect to the basis $\{y^{*N-i}z^{*i}\}_{i=0}^N$ the matrix representing a is $\text{diag}(N, N-2, \dots, -N)$ and the matrix representing a^* is irreducible tridiagonal. Therefore a, a^* act on V as a Leonard pair of Krawtchouk type. \square

Theorem 6.4 *Let V denote a vector space over \mathbb{F} with dimension $N + 1$ and let A, A^* denote a Leonard pair on V that has Krawtchouk type. Then there exists an L -module structure on V and a pair of normalized semisimple elements of L that generate L and act on V as A, A^* . The L -module V is isomorphic to the L -module $\text{Hom}_N(A)$ from Section 3.*

Proof. We assume $N \geq 2$; otherwise the result is routine. Let $\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ denote a Leonard system on V associated with A, A^* . Let $\{\theta_i\}_{i=0}^N$ (resp. $\{\theta_i^*\}_{i=0}^N$) denote the eigenvalue sequence (resp. dual eigenvalue sequence) of Φ . By construction we may assume

$$\theta_i = N - 2i, \quad \theta_i^* = N - 2i, \quad i = 0, 1, \dots, N. \quad (69)$$

Claim 1. A, A^* satisfy both

$$[A, [A, [A, A^*]]] = 4[A, A^*], \quad (70)$$

$$[A^*, [A^*, [A^*, A]]] = 4[A^*, A]. \quad (71)$$

Proof. We first show (70). Let C denote the left-hand side of (70) minus the right-hand side of (70). Observe that

$$C = A^3 A^* - 3A^2 A^* A + 3A A^* A^2 - A^* A^3 - 4(AA^* - A^* A). \quad (72)$$

We show $C = 0$. Since $I = \sum_{i=0}^N E_i$, it suffices to show $E_i C E_j = 0$ for $i, j = 0, 1, \dots, N$. Let i, j be given. Expand $E_i C E_j$ using (72), and simplify using $E_i A = \theta_i E_i$ and $A E_j = \theta_j E_j$ to find

$$E_i C E_j = E_i A^* E_j (\theta_i - \theta_j + 2)(\theta_i - \theta_j - 2)(\theta_i - \theta_j).$$

Observe that $E_i A^* E_j = 0$ if $|i - j| > 1$, $\theta_i - \theta_j + 2 = 0$ if $i - j = 1$, $\theta_i - \theta_j - 2 = 0$ if $j - i = 1$, and $\theta_i - \theta_j = 0$ if $i = j$. In all cases $E_i C E_j = 0$. We have shown $E_i C E_j = 0$ for $i, j = 0, 1, \dots, N$. Therefore $C = 0$ so (70) holds. The proof of (71) is similar. We have shown Claim 1.

Claim 2. There exists $p \in \mathbb{F}$ such that both

$$[A, [A, A^*]] = 4(2p - 1)A + 4A^*, \quad (73)$$

$$[A^*, [A^*, A]] = 4(2p - 1)A^* + 4A. \quad (74)$$

Proof. Let \mathcal{D} denote the subalgebra of $\text{End}(V)$ generated by A . Since A is multiplicity-free,

$$\mathcal{D} = \{y \in \text{End}(V) \mid [y, A] = 0\}.$$

The element $[A, [A, A^*]] - 4A^*$ commutes with A by (70), so this element is contained in \mathcal{D} . Therefore there exist scalars $\{\alpha_i\}_{i=0}^N$ in \mathbb{F} such that

$$[A, [A, A^*]] - 4A^* = \sum_{i=0}^N \alpha_i A^i. \quad (75)$$

We show $\alpha_i = 0$ for $3 \leq i \leq N$. Suppose not, and let $k = \max\{i \mid 3 \leq i \leq N, \alpha_i \neq 0\}$. In equation (75), multiply each side on the left by E_k^* and on the right by E_0^* . Expand the result to find

$$E_k^*(A^2A^* - 2AA^*A + A^*A^2 - 4A^*)E_0^* = \sum_{i=0}^k \alpha_i E_k^* A^i E_0^*. \quad (76)$$

Using Lemma 5.11(i) we find that the left-hand side of (76) is 0 and the right-hand side of (76) equals $\alpha_k E_k^* A^k E_0^*$. Therefore $\alpha_k E_k^* A^k E_0^* = 0$. Recall $\alpha_k \neq 0$ by construction and $E_k^* A^k E_0^* \neq 0$ by Lemma 5.11(i). Therefore $\alpha_k E_k^* A^k E_0^* \neq 0$, for a contradiction. We have shown $\alpha_i = 0$ for $3 \leq i \leq N$.

Next we show $\alpha_2 = 0$. So far we have

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = \alpha_0 I + \alpha_1 A + \alpha_2 A^2.$$

In this equation, multiply each side on the left by E_2^* and on the right by E_0^* to find

$$E_2^*(A^2A^* - 2AA^*A + A^*A^2 - 4A^*)E_0^* = E_2^*(\alpha_0 I + \alpha_1 A + \alpha_2 A^2)E_0^*. \quad (77)$$

In (77) we evaluate the terms in the left-hand side. To aid in this evaluation we make some comments. Using $A^* = \sum_{i=0}^N \theta_i^* E_i^*$ and Definition 5.4(v) we find

$$E_2^* A A^* A E_0^* = \theta_1^* E_2^* A E_1^* A E_0^*. \quad (78)$$

Using $I = \sum_{i=0}^N E_i^*$ and Definition 5.4(v) we find

$$E_2^* A^2 E_0^* = E_2^* A I A E_0^* = E_2^* A E_1^* A E_0^*. \quad (79)$$

Combining (78) and (79) we find $E_2^* A A^* A E_0^* = \theta_1^* E_2^* A^2 E_0^*$. By these comments the left-hand side of (77) is equal to $(\theta_0^* - 2\theta_1^* + \theta_2^*) E_2^* A^2 E_0^*$. This is 0 since $\theta_0^* - 2\theta_1^* + \theta_2^* = 0$ by (69). The right-hand side of (77) is equal to $\alpha_2 E_2^* A^2 E_0^*$ by Lemma 5.11(i). Therefore $\alpha_2 E_2^* A^2 E_0^* = 0$. We have $E_2^* A^2 E_0^* \neq 0$ by Lemma 5.11(i) so $\alpha_2 = 0$.

Next we show $\alpha_0 = 0$. So far we have

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = \alpha_0 I + \alpha_1 A. \quad (80)$$

In this equation we multiply each side on the left by E_0 and on the right by E_0^* . Simplify the result using Lemma 5.14(ii) and then Lemma 5.14(i). Simplify the result of that using $E_0 E_0^* \neq 0$ to find

$$(a_0 - \theta_0)(\theta_1 - \theta_0)(\theta_0^* - \theta_1^*) - 4\theta_0^* = \alpha_0 + \alpha_1 \theta_0. \quad (81)$$

In equation (80), multiply each side on the left by E_N^* and on the right by E_N . Simplify the result using Lemma 5.15(ii) and then Lemma 5.15(i). Simplify the result of that using $E_N^* E_N \neq 0$ to find

$$(\theta_N^* - a_N^*)(\theta_{N-1} - \theta_N)^2 - 4\theta_N^* = \alpha_0 + \alpha_1 \theta_N. \quad (82)$$

View (81), (82) as a linear system of equations in the unknowns α_0, α_1 . The coefficient matrix is nonsingular since $\theta_0 \neq \theta_N$. Solving this system for α_0 and simplifying the result using (68), (69) we find $\alpha_0 = 0$.

So far we have $\alpha_i = 0$ for $2 \leq i \leq N$ and $\alpha_0 = 0$. Therefore (75) becomes

$$[A, [A, A^*]] - 4A^* = \alpha_1 A. \quad (83)$$

Interchanging the roles of A and A^* in our argument so far, we see that there exists $\alpha_1^* \in \mathbb{F}$ such that

$$[A^*, [A^*, A]] - 4A = \alpha_1^* A^*. \quad (84)$$

We show $\alpha_1 = \alpha_1^*$. In (83), take the commutator of each term with A^* to find

$$[[A, [A, A^*]], A^*] = \alpha_1 [A, A^*]. \quad (85)$$

Similarly using (84),

$$[A, [A^*, [A^*, A]]] = \alpha_1^* [A, A^*]. \quad (86)$$

In (85) and (86), the left-hand sides are equal by the Jacobi identity, so $(\alpha_1 - \alpha_1^*)[A, A^*] = 0$. Observe that $[A, A^*] \neq 0$; otherwise each of $\{E_i\}_{i=0}^N$ commutes with A^* in view of (56), contradicting Definition 5.4(iv). Therefore $\alpha_1 = \alpha_1^*$. Now define $p \in \mathbb{F}$ such that $4(2p - 1) = \alpha_1$. Then (83), (84) become (73), (74). We have shown Claim 2.

Claim 3. We have $p \neq 0$ and $p \neq 1$.

Proof. By Claim 2,

$$0 = A^2 A^* - 2AA^*A + A^*A^2 - 4(2p - 1)A - 4A^*, \quad (87)$$

$$0 = A^{*2}A - 2A^*AA^* + AA^{*2} - 4(2p - 1)A^* - 4A. \quad (88)$$

In these equations, multiply each side on the left by E_0^* and on the right by E_0^* . Simplify the result using $E_0^*A^* = \theta_0^*E_0^*$ and $A^*E_0^* = \theta_0^*E_0^*$ to find

$$0 = \theta_0^*E_0^*A^2E_0^* - E_0^*AA^*AE_0^* - 2(2p - 1)E_0^*AE_0^* - 2\theta_0^*E_0^*, \quad (89)$$

$$0 = E_0^*AE_0^* + (2p - 1)\theta_0^*E_0^*. \quad (90)$$

Eliminating $E_0^*AE_0^*$ from (89) using (90) we find

$$0 = \theta_0^*E_0^*A^2E_0^* - E_0^*AA^*AE_0^* + 8p(p - 1)\theta_0^*E_0^*. \quad (91)$$

Pick a Φ -basis $\{v_i\}_{i=0}^N$ for V , and identify each element of $\text{End}(V)$ with the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents it with respect to $\{v_i\}_{i=0}^N$. In (91) we compute the $(0, 0)$ -entry of each side, and find

$$0 = (\theta_0^* - \theta_1^*)A_{01}A_{10} + 8p(p - 1)\theta_0^*.$$

Each of A_{01} , A_{10} is nonzero since A is irreducible tridiagonal, so $p(p - 1) \neq 0$. We have shown Claim 3.

Comparing Claims 2 and 3 with Lemma 2.13 we get all the assertions of the theorem except the last one. To get the last assertion we invoke Lemma 3.6. Let a, a^* denote a pair of normalized semisimple elements of L that generate L and act on V as A, A^* . By the comment above (14) we may assume that the basis e, h, f for L is related to a, a^* according to (14). Pick a Φ -basis $\{v_i\}_{i=0}^N$ for V . We show that this basis satisfies condition (ii) of Lemma 3.6. By construction $h.v_i = (N - 2i)v_i$ for $i = 0, 1, \dots, N$. We now show $e.v_0 = 0$. We have $a^*.v_0 = Nv_0$ and $a^*.v_1 = (N - 2)v_1$. We also have $a.v_0 \in \text{Span}\{v_0, v_1\}$, so there

exist scalars ξ, η in \mathbb{F} such that $a.v_0 = \xi v_0 + \eta v_1$. Using these comments we apply (11) to v_0 and find $\xi = (1 - 2p)N$. By Lemma 2.16(i),

$$e = \frac{2a + 2(2p - 1)a^* - [a, a^*]}{8(1 - p)}.$$

In this equation we apply each term to v_0 . Simplifying the result using the above comments we routinely find $e.v_0 = 0$. Next we show $f.v_N = 0$. We have $a^*.v_N = -Nv_N$ and $a^*.v_{N-1} = (2 - N)v_{N-1}$. We also have $a.v_N \in \text{Span}\{v_N, v_{N-1}\}$, so there exist scalars μ, ν in \mathbb{F} such that $a.v_N = \mu v_N + \nu v_{N-1}$. Using these comments we apply (11) to v_N and find $\mu = (2p - 1)N$. By Lemma 2.16(i),

$$f = \frac{2a + 2(2p - 1)a^* + [a, a^*]}{8p}.$$

In this equation we apply each term to v_N . Simplifying the result using the above comments we routinely find $f.v_N = 0$. We have shown that V satisfies condition (ii) of Lemma 3.6. By Lemma 3.6 the L -module V is isomorphic to the L -module $\text{Hom}_N(\mathcal{A})$. \square

We have been discussing the Leonard pairs of Krawtchouk type. For a study of general Leonard pairs we recommend the survey paper [27].

7 Acknowledgements

The authors thank Erik Koelink and Tom Koornwinder for sending us detailed historical information concerning the Krawtchouk polynomials and \mathfrak{sl}_2 .

References

- [1] G. E. Andrews, R. Askey, R. Roy, Special Functions, Encyclopedia of Mathematics and Its Applications 71, Cambridge University Press, 1999.
- [2] R. Askey, Evaluation of Sylvester type determinants using orthogonal polynomials, in: Advances in Analysis, Proceedings of the 4th International ISAAC Congress, York University, Toronto, Canada, 2003, pp. 1–16.
- [3] E. Bannai, T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings, London, 1984.
- [4] P. Delsarte, An algebraic approach to the association schemes of coding theory, Philips Res. Rep. Supp. 10 (1973).
- [5] G. K. Eagleson, A duality relation for discrete orthogonal systems, J. Studia Sci. Math. Hungar. 3 (1968) 127–136.
- [6] P. Feinsilver, Lie algebras and recurrence relations I, Acta Appl. Math. 13 (1988) 291–333.
- [7] P. Feinsilver, Representations of $\mathfrak{sl}(2)$ in the Boolean lattice, and the Hamming and Johnson schemes, preprint, [arXiv:1102.0368](https://arxiv.org/abs/1102.0368).
- [8] P. Feinsilver, J. Kocik, Krawtchouk polynomials and Krawtchouk matrices, in: Recent Advances in Applied Probability, Edited by Ricardo Baeza-Yates, etc. Springer Science+Business Media, Inc. 2005, pp. 115–141; [arXiv:quant-ph/0702073](https://arxiv.org/abs/quant-ph/0702073).

- [9] P. Feinsilver and R. Schott, Algebraic Structures and Operator Calculus III, Representations of Lie groups, Kluwer Academic Publishers, 1996.
- [10] Ya. I. Granovskii, I. M. Lutzenko, A. S. Zhedanov, Mutual integrability, quadratic algebras, and dynamical symmetry, *Ann. Physics* 217 (1992) 1–20.
- [11] J. Go, The Terwilliger algebra of the hypercube, *European J. Combin.* 23 (2002) 399–429.
- [12] J. E. Humphreys, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Math. 9, Springer, New York, 1972.
- [13] T. Ito, K. Tanabe, P. Terwilliger, Some algebra related to P - and Q -polynomial association schemes, in: Codes and Association Schemes (Piscataway NJ, 1999), Amer. Math. Soc., Providence RI, 2001, pp. 167–192; [arXiv:math/0406556](#).
- [14] M. Kac, Random walk and the theory of Brownian motion, *Amer. Math. Monthly* 54 (1947) 369–391.
- [15] R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric Orthogonal Polynomials and Their q -Analogues, Springer monograph series, 2010, Springer.
- [16] H. T. Koelink, Askey-Wilson polynomials and the quantum $SU(2)$ group; survey and applications, *Acta Appl. Math.* 44 (1996) 295–352.
- [17] T. H. Koornwinder, Krawtchouk polynomials, a unification of two different group theoretic interpretations, *SIAM J. Math. Anal.* 13 (1982) 1011–1023.
- [18] D. Leonard, Orthogonal polynomials, duality and association schemes, *SIAM J. Math. Anal.* 13 (1982) 656–663.
- [19] W. Miller, Lie theory and difference equations I, *J. Math. Anal. Appl.* 28 (1969) 383–399.
- [20] H. Rosengren, An elementary approach to $6j$ -symbols (classical, quantum, rational, trigonometric, and elliptic), *Ramanujan J.* 13 (2007) 131–166; [arXiv:math/0312310](#).
- [21] N. J. A. Sloane, An introduction to association schemes and coding theory, in: Theory and Application of Special Functions, Edited by R. Askey, Proc. Sem. Madison 1975, Academic Press, New York, 1975, pp. 225–260.
- [22] J. J. Sylvester, *Nouvelles Annales de Mathématiques*, XIII (1854), 305, Reprinted in *Collected Mathematical Papers*, vol. II, 28.
- [23] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other, *Linear Algebra Appl.* 330 (2001) 149–203; [arXiv:math/0406555](#).
- [24] P. Terwilliger, Introduction to Leonard pairs, OPSFA Rome 2001, *J. Comput. Appl. Math.* 153 (2003) 463–475.
- [25] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; the TD - D canonical form and the LB - UB canonical form, *J. Algebra* 291 (2005) 1–45; [arXiv:math/0304077](#).
- [26] P. Terwilliger, The subconstituent algebra of an association scheme I, *J. Algebraic Combin.* 1 (1992) 363–388.
- [27] P. Terwilliger, An algebraic approach to the Askey scheme of orthogonal polynomials, Orthogonal polynomials and special functions, Lecture Notes in Math., 1883, Springer, Berlin, 2006, pp. 255–330; [arXiv:math/0408390](#).
- [28] P. Terwilliger, R. Vidunas, Leonard pairs and the Askey-Wilson relations, *J. Algebra Appl.* 3 (2004) 411–426; [arXiv:math/0305356](#).

- [29] N. J. Vilenkin, A. U. Klimyk, Representation of Lie Groups and Special Functions. Vol. 1. Simplest Lie groups, special functions and integral transforms, Mathematics and its Applications (Soviet Series), 72. Kluwer Academic Publishers Group, Dordrecht, 1991.

Kazumasa Nomura
Professor Emeritus
Tokyo Medical and Dental University
Kohnodai, Ichikawa, 272-0827 Japan
email: knomura@pop11.odn.ne.jp

Paul Terwilliger
Department of Mathematics
University of Wisconsin
480 Lincoln Drive
Madison, Wisconsin, 53706 USA
email: terwilli@math.wisc.edu

Keywords. Lie algebra, Krawtchouk polynomial, Leonard pair
2010 Mathematics Subject Classification. 33C05